Computational Treatment of First Order Delay Differential Equations Using Hybrid Extended Second Derivative Block Backward Differentiation Formulae

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***Abstract*—This paper considers the computational solution of first order delay differential equations (DDEs) using hybrid extended second derivative backward differentiation formulae method in block form without the implementation of interpolation techniques in estimating the delay term. By matrix inversion approach, the discrete schemes were obtained through the linear multistep collocation approach from the continuous form of each step number which after implementation strongly revealed the convergence and region of absolute stability of the proposed method. Computational results are presented and compared to the exact solutions and other existing method to demonstrate its efficiency and accuracy.**

***Keywords*—backward differentiation formulae; extended future points; first order delay differential equations; hybrid block method; off-grid point. Mathematical Subject Classification: 34K28; 65F30**

I. INTRODUCTION

Many years ago, the study of delay differential equations in the numerical solution of multistep block method has gained serious attention among the scholars. Initially these methods were used in solving ODE and have revealed its less advantage in obtaining good accurate results. Delay differential equations (DDEs) is one of the mathematical models in which its unknown function depends not only on current value but also on the past value which is called a delay term. The applications of DDEs can be seen in many real life situations involving celestial and quantum mechanics, nuclear and theoretical physics, astrophysics, quantum chemistry, molecular dynamics, engineering, medicine and economic dynamics and control.

In this research work, we look forward to obtaining the computational solutions of the first order delay differential equations (DDEs) of the form as developed by [1]

$$\begin{aligned} &x'(t) = f(t, x(t), x(t-\tau)), \quad \text{for } t > t_0, \tau > 0 \\ &x(t) = \alpha(t), \text{for } t \le t_0 \end{aligned} \tag{1}$$

where $\alpha(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument and $x(t-\tau)$ is the solution of the delay argument using hybrid extended second derivative block backward differentiation formulae methods.

In [2] linear multistep approach was used in solving delay differential equations with the aid of interpolation techniques in estimating the delay argument. [2] adopted linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays with Nordsieck's interpolation technique. [4] applied implicit 2-point Block Backward Differential Formulae to solve a set of delay differential equations with the help of interpolation technique in approximating the delay term. [5] implemented block simpson's methods in solving first order delay differential equations without using any interpolation techniques in examining the delay term. [6] developed a class of sixth order hybrid extended block backward differentiation formulae for computational solutions of first order delay differential equations without the use of interpolation techniques in investigating the delay argument. [7] applied construction of extended second derivative block backward differentiation formulae for numerical solutions of first order delay differential equations without the use of any interpolation technique in evaluating the delay term. [8] implemented third derivative block backward differentiation formulae for numerical solutions of first order delay differential equations without interpolation techniques in investigating the delay argument. [9] solved first order delay differential equations using multiple off-grid hybrid block simpson's methods without the application of any interpolation technique in estimating the delay term. [10] investigated the numerical solution of first order DDEs using second derivative block backward differentiation formulae without the introduction of interpolation techniques in finding the delay argument. [11] solved first order DDEs using extrapolated block backward differentiation formulae for efficiency of the numerical solution without the introduction of interpolation techniques in finding the delay argument. [12] formulated a new hybrid block adams moulton methods with two and three off grid points for solution of first order delay differential equations without the

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implementation of interpolation techniques in evaluating the delay term.

One of the challenges in using these interpolation techniques by [13] is that the numerical methods to be applied in solving DDEs should be at least the same with the order of the interpolating polynomials which is very hard to carry out; otherwise, the accuracy of the method will not be preserved. It is very crucial that in the evaluation of the delay term, applying a tested and reliable formula shall be highly recommended.

In order to overcome the challenges caused in using the interpolation techniques in evaluating the delay term; we applied idea of the sequence constructed by [14] which we merge into the first order delay differential equations. Then we implemented hybrid extended second derivative block backward differentiation formulae methods to solve some first order delay differential equations containing the evaluated delay argument to improve the performance of the existing methods studied by [7,10,14] in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at fixed step size v. The improve performance shown by this method proved its superiority over the other existing methods.

II. FORMULATION PROCEDURE

A. Formulation of Hybrid Extended Second Derivative Backward Differentiation Formulae Method

In [15], the k-step generalized Backward Differentiation Formulae Methods was constructed as

$$y_{c+w}=\sum_{r=0}^{w}\alpha_r(x)y_{c+r}+v\beta_r(x)f(x_r,y(x_r)) \qquad (2)$$

We developed the equation for second derivative BDF using the basis function,

$$y(x)=\sum_{r=0}^{w}\alpha_r\left(\frac{x-x_C}{v}\right)^r \qquad (3)$$

Equation (3) is then interpolated at points x_{c+r} , $r=0,1,2,...,j-1,$

while $y'(x)$ and $y''(x)$ are collocated at point x_{c+w}. The system of equations obtained is evaluated for $\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_{w+1}$ which is substituted back into (3) to obtain the continuous second derivative backward differentiation formula of the form,

$$y_{c+w}=\sum_{r=0}^{w}\alpha_r(x)y_{c+r}+v\beta_r(x)f(x_r,y(x_r)+v^2\gamma_r(x)g(x_r,y(x_r)$$

(4)

Following the same approach of using (3), the extended second derivative BDF is expressed as

$$y_{c+w}=\sum_{r=0}^{w}\alpha_r(x)y_{c+r}+v\beta_r(x)f(x_r,y(x_r)+v^2\gamma_r(x)g(x_r,y(x_r)+v^2\delta_r(x)q(x_r,y(x_r)$$

(5)

where

$$y_{c+r}=y(x_c+rv), f_{c+w}\equiv f(x_c+rv,y(x_c+rv),y'(x_c+rv)),$$

$g_{c+w}\equiv\left.\dfrac{df}{dx}\right|_{y=y_{c+w}}^{x=x_{c+w}}$ while $q_{c+w}\equiv\left.\dfrac{df}{dx}\right|_{y=y_{c+w}}^{x=x_{c+w}}$ is the extension and $\alpha_r(x)$, $\beta_r(x)$, $\gamma_r(x)$, $\delta_r(x)$ are the continuous coefficients of the method defined as

$$\alpha_r(x)=\sum_{p=0}^{j+s-1}\alpha_{r,p+1}x^p \qquad (6)$$

for $r=\{0,1,...,j-1\}$

$$v\beta_r(x)=\sum_{p=0}^{j+s-1}v\beta_{r,p+1}x^p \qquad (7)$$

for $r=\{0,1,...,s-1\}$

$$v^2\gamma_r(x)=\sum_{p=1}^{w}v^2\gamma_{r,p+1}x^p \qquad (8)$$

for $r=\{0,1,...,s-1\}$

$$v^2\delta_r(x)=\sum_{p=1}^{w}v^2\gamma_{r,p+1}x^p \qquad (9)$$

for $r=\{0,1,...,s-1\}$

where $x_0,\ \ldots,x_{s-1}$ are the s collocation points, x_{c+r} , $r=0,1,2,...,j-1$ are the j arbitrarily chosen interpolation points and v is the fixed step width and k is the step number.

To get continuous coefficients $\alpha_r(x)$, $\beta_r(x)$, $\gamma_r(x)$ and $\delta_r(x)$ of the propose method, we constructed a matrix equation of the form

$$NP=\mathrm{I} \qquad (10)$$

where I represents the unit matrix of dimension $(j+s)\times(j+s)$ and N and P are matrices presented as

$$N=\begin{pmatrix}\alpha_{0,1} & \alpha_{1,1} & \cdot & \alpha_{j-1,1} & v\beta_{0,1} & \cdot & v\beta_{s-1,1} & v^2\gamma_{0,1} & \cdot & v^2\gamma_{s-1,1} & v^2\delta_{0,1} & \cdot & v^2\delta_{s-1,1}\\ \alpha_{0,2} & \alpha_{1,2} & \cdot & \alpha_{j-1,2} & v\beta_{0,2} & \cdot & v\beta_{s-1,2} & v^2\gamma_{0,2} & \cdot & v^2\gamma_{s-1,2} & v^2\delta_{0,2} & \cdot & v^2\delta_{s-1,2}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \alpha_{0,r+s} & \alpha_{1,r+s} & \cdot & \alpha_{j-1,r+s} & b\beta_{0,r+s} & \cdot & v\beta_{s-1,r+s} & v^2\gamma_{0,r+s} & \cdot & v^2\gamma_{s-1,r+s} & v^2\delta_{0,r+s} & \cdot & v^2\delta_{s-1,r+s}\end{pmatrix} \qquad (11)$$

$$P=\begin{pmatrix}1 & x_c & x_c^2 & x_c^3 & x_c^4 & \cdot\ \cdot & x_c^{j+s-1}\\ 1 & x_{c+1} & x_{c+1}^2 & x_{c+1}^3 & x_{c+1}^4 & \cdot\ \cdot & x_{c+1}^{j+s-1}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot\\ 1 & x_{c+j-1} & x_{c+j-1}^2 & x_{c+j-1}^3 & x_{c+j-1}^4 & \cdot\ \cdot & x_{c+j-1}^{j+s-1}\\ 0 & 1 & 2x_c & 3x_c^2 & 4x_c^3 & \cdot\ \cdot & (j+s-1)x_c^{j+s-2}\\ 0 & 1 & 2x_{c+1} & 3x_{c+1}^2 & 4x_{c+1}^3 & \cdot\ \cdot & (j+s-1)x_{c+1}^{j+s-2}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot\\ 0 & 1 & 2x_{c+j-1} & 3x_{c+j-1}^2 & 4x_{c+j-1}^3 & \cdot\ \cdot & (j+s-1)x_{c+j-1}^{j+s-2}\\ 0 & 0 & 2 & 6x_c & 12x_c^2 & \cdot\ \cdot & (j+s-1)(j+s-2)x_c^{j+s-3}\\ 0 & 0 & 2 & 6x_{c+1} & 12x_{c+1}^2 & \cdot\ \cdot & (j+s-1)(j+s-2)x_{c+1}^{j+s-3}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot\\ 0 & 0 & 2 & 6x_{c+j-1} & 12x_{c+j-1}^2 & \cdot\ \cdot & (j+s-1)(j+s-2)x_{c+j-1}^{j+s-3}\end{pmatrix} \qquad (12)$$

From (10) the columns of $N=P^{-1}$ give the continuous coefficients of the continuous scheme (5).

B. Construction of HESDBBDF Method with Integrated Three Off-grids Extended Future Points and One Extended Future Point for $k=2$

Here, we integrated three off-grids extended future points at $x=x_{c+\frac{9}{4}},x=x_{c+\frac{5}{2}},x=x_{c+\frac{11}{4}}$ and one extended future point at $x = x_{c+3}$ as collocation points, thus the interpolation points, $j = 2$ and the collocation points $s = 6$ are considered, therefore, (5) becomes

$$y(x)=\alpha_0(x)y_c+\alpha_1(x)y_{c+1}+v[\beta_2(x)f_{c+2}+v\gamma_2(x)g_{c+2}+v\delta_{\frac{9}{4}}(x)g_{c+\frac{9}{4}}+v\eta_{\frac{5}{2}}(x)g_{c+\frac{5}{2}}+v\varepsilon_{\frac{11}{4}}(x)g_{c+\frac{11}{4}}+v\omega_3(x)q_{c+3}] \tag{13}$$

The matrix P in (10) becomes

$$P=\begin{pmatrix}
1 & x_c & x_c^2 & x_c^3 & x_c^4 & x_c^5 & x_c^6 & x_c^7 \\
1 & x_c+v & (x_c+v)^2 & (x_c+v)^3 & (x_c+v)^4 & (x_c+v)^5 & (x_c+v)^6 & (x_c+v)^7 \\
0 & 1 & 2x_c+4v & 3(x_c+2v)^2 & 4(x_c+2v)^3 & 5(x_c+2v)^4 & 6(x_c+2v)^5 & 7(x_c+2v)^6 \\
0 & 0 & 2 & 6x_c+12v & 12(x_c+2v)^2 & 20(x_c+2v)^3 & 30(x_c+2v)^4 & 42(x_c+2v)^5 \\
0 & 0 & 2 & 6x_c+\frac{27}{2}v & 12\left(x_c+\frac{9}{4}v\right)^2 & 20\left(x_c+\frac{9}{4}v\right)^3 & 30\left(x_c+\frac{9}{4}v\right)^4 & 42\left(x_c+\frac{9}{4}v\right)^5 \\
0 & 0 & 2 & 6x_c+15v & 12\left(x_c+\frac{5}{2}v\right)^2 & 20\left(x_c+\frac{5}{2}v\right)^3 & 30\left(x_c+\frac{5}{2}v\right)^4 & 42\left(x_c+\frac{5}{2}v\right)^5 \\
0 & 0 & 2 & 6x_c+\frac{33}{2}v & 12\left(x_c+\frac{11}{4}v\right)^2 & 20\left(x_c+\frac{11}{4}v\right)^3 & 30\left(x_c+\frac{11}{4}v\right)^4 & 42\left(x_c+\frac{11}{4}v\right)^5 \\
0 & 0 & 2 & 6x_c+18v & 12(x_c+3v)^2 & 20(x_c+3v)^3 & 30(x_c+3v)^4 & 42(x_c+3v)^5
\end{pmatrix} \tag{14}$$

The inverse of the matrix $N = P^{-1}$ is evaluated using Maple 18 from which the continuous scheme is obtained using (5), evaluating and simplifying it at $x = x_{c+2}, x = x_{c+\frac{9}{4}}, x = x_{c+\frac{5}{2}}, x = x_{c+\frac{11}{4}}, x = x_{c+3}$ and its derivative at $x = x_{c+1}$, the following discrete schemes are obtained

$$\begin{aligned}
y_{c+1}&=\frac{3195957}{40880}v^2g_{c+2}+\frac{1204859}{6132}v^2g_{c+\frac{5}{2}}-\frac{4365586}{22995}v^2g_{c+\frac{9}{4}}-\frac{105674}{1095}v^2g_{c+\frac{11}{4}}+\frac{6833377}{367920}v^2q_{c+3}+\frac{34273}{4088}vf_{c+1}-\frac{30185}{4088}vf_{c+2}+y_c\\
y_{c+2}&=-\frac{799}{34273}y_c+\frac{35072}{34273}y_{c+1}+\frac{33474}{34273}vf_{c+2}-\frac{1591042}{514095}v^2g_{c+2}+\frac{3420416}{514095}v^2g_{c+\frac{9}{4}}-\frac{3376096}{514095}v^2g_{c+\frac{5}{2}}+\frac{537344}{171365}v^2g_{c+\frac{11}{4}}-\frac{20292}{34273}v^2q_{c+3}\\
y_{c+\frac{9}{4}}&=-\frac{409195}{17547776}y_c+\frac{17956971}{17547776}y_{c+1}+\frac{21525525}{17547776}vf_{c+2}-\frac{108033821}{35095552}v^2g_{c+2}+\frac{14640587}{2193472}v^2g_{c+\frac{9}{4}}-\frac{115404915}{17547776}v^2g_{c+\frac{5}{2}}+\frac{6885593}{2193472}v^2g_{c+\frac{11}{4}}-\frac{5199671}{8773888}v^2q_{c+3}\\
y_{c+\frac{5}{2}}&=-\frac{1599}{68546}y_c+\frac{70145}{68546}y_{c+1}+\frac{50610}{34273}vf_{c+2}-\frac{3352191}{1096736}v^2g_{c+2}+\frac{1387699}{205638}v^2g_{c+\frac{9}{4}}-\frac{3609059}{548368}v^2g_{c+\frac{5}{2}}+\frac{215441}{68546}v^2g_{c+\frac{11}{4}}-\frac{1952125}{3290208}v^2q_{c+3}\\
y_{c+\frac{11}{4}}&=-\frac{409493}{17547776}y_c+\frac{17957269}{17547776}y_{c+1}+\frac{30299115}{17547776}vf_{c+2}-\frac{1597733137}{526433280}v^2g_{c+2}+\frac{224660513}{32902080}v^2g_{c+\frac{9}{4}}-\frac{1720324991}{263216640}v^2g_{c+\frac{5}{2}}+\frac{34581701}{10967360}v^2g_{c+\frac{11}{4}}-\frac{3258717}{5483680}v^2q_{c+3}\\
y_{c+3}&=-\frac{800}{34273}y_c+\frac{35073}{34273}y_{c+1}+\frac{67746}{34273}vf_{c+2}-\frac{309899}{102819}v^2g_{c+2}+\frac{3552352}{514095}v^2g_{c+\frac{9}{4}}-\frac{1111732}{171365}v^2g_{c+\frac{5}{2}}+\frac{1653856}{514095}v^2g_{c+\frac{11}{4}}-\frac{303517}{514095}v^2q_{c+3}
\end{aligned} \tag{15}$$

C. Construction of HESDBBDF Method with Integrated Two Off-grids Extended Future Points and One Extended Future Point for $k = 3$

In this case, we integrated two off-grids extended future points at $x = x_{c+\frac{7}{2}}, x = x_{c+\frac{15}{4}}$ and one extended future point at $x = x_{c+4}$ as collocation points, thus the interpolation points, $j = 3$ and the collocation points $s = 5$ are considered, therefore, (5) becomes

$$y(x)=\alpha_0(x)y_c+\alpha_1(x)y_{c+1}+\alpha_2(x)y_{c+2}+v[\beta_3(x)f_{c+3}+v\gamma_3(x)g_{c+3}+v\eta_{\frac{7}{2}}(x)g_{c+\frac{7}{2}}+v\varepsilon_{\frac{15}{4}}(x)g_{c+\frac{15}{4}}+v\omega_4(x)q_{c+4}] \tag{16}$$

The matrix P in (10) becomes

$$P=\begin{pmatrix}
1 & x_c & x_c^2 & x_c^3 & x_c^4 & x_c^5 & x_c^6 & x_c^7 \\
1 & x_c+v & (x_c+v)^2 & (x_c+v)^3 & (x_c+v)^4 & (x_c+v)^5 & (x_c+v)^6 & (x_c+v)^7 \\
1 & x_c+2v & (x_c+2v)^2 & (x_c+2v)^3 & (x_c+2v)^4 & (x_c+2v)^5 & (x_c+2v)^6 & (x_c+2v)^7 \\
0 & 1 & 2x_c+6v & 3(x_c+3v)^2 & 4(x_c+3v)^3 & 5(x_c+3v)^4 & 6(x_c+3v)^5 & 7(x_c+3v)^6 \\
0 & 0 & 2 & 6x_c+18v & 12(x_c+3v)^2 & 20(x_c+3v)^3 & 30(x_c+3v)^4 & 42(x_c+3v)^5 \\
0 & 0 & 2 & 6x_c+21v & 12\left(x_c+\frac{7}{2}v\right)^2 & 20\left(x_c+\frac{7}{2}v\right)^3 & 30\left(x_c+\frac{7}{2}v\right)^4 & 42\left(x_c+\frac{7}{2}v\right)^5 \\
0 & 0 & 2 & 6x_c+\frac{45}{2}v & 12\left(x_c+\frac{15}{4}v\right)^2 & 20\left(x_c+\frac{15}{4}v\right)^3 & 30\left(x_c+\frac{15}{4}v\right)^4 & 42\left(x_c+\frac{15}{4}v\right)^5 \\
0 & 0 & 2 & 6x_c+24v & 12(x_c+4v)^2 & 20(x_c+4v)^3 & 30(x_c+4v)^4 & 42(x_c+4v)^5
\end{pmatrix} \tag{17}$$

The inverse of the matrix $N = P^{-1}$ is evaluated using Maple 18 from which the continuous scheme is obtained using (5), evaluating and simplifying it at $x = x_{c+3}, x = x_{c+\frac{7}{2}}, x = x_{c+\frac{15}{4}}, x = x_{c+4}$ and its derivative at $x = x_{c+1}$ and $x = x_{c+2}$ the following discrete schemes are obtained

$$\begin{aligned}
y_{c+1}&=\frac{626087947}{336161310}v^2g_{c+3}-\frac{680587916}{168080655}v^2g_{c+\frac{7}{2}}+\frac{630800768}{168080655}v^2g_{c+\frac{15}{4}}-\frac{688630997}{672322620}v^2q_{c+4}-\frac{56631783}{89643016}vf_{c+1}-\frac{46703841}{89643016}vf_{c+3}-\frac{855788}{11205377}y_c+\frac{12061165}{11205377}y_{c+2}\\
y_{c+2}&=\frac{2573430208}{541733385}v^2g_{c+3}-\frac{4603388288}{541733385}v^2g_{c+\frac{7}{2}}+\frac{1356812288}{180577795}v^2g_{c+\frac{15}{4}}-\frac{1072270634}{541733385}v^2q_{c+4}+\frac{113263566}{36115559}vf_{c+2}-\frac{79330800}{36115559}vf_{c+3}-\frac{2182793}{36115559}y_c+\frac{38298352}{36115559}y_{c+1}\\
y_{c+3}&=\frac{106888}{18877261}y_c-\frac{1653939}{18877261}y_{c+1}+\frac{20424312}{18877261}y_{c+2}+\frac{17437098}{18877261}vf_{c+3}-\frac{75518022}{94386305}v^2g_{c+3}+\frac{105233472}{94386305}v^2g_{c+\frac{7}{2}}-\frac{6899712}{7260485}v^2g_{c+\frac{15}{4}}-\frac{23054436}{94386305}v^2q_{c+4}\\
y_{c+\frac{7}{2}}&=\frac{6938495}{1208144704}y_c-\frac{1671544}{18877261}y_{c+1}+\frac{1308185025}{1208144704}y_{c+2}+\frac{859557615}{604072352}vf_{c+3}-\frac{2642247503}{3624434112}v^2g_{c+3}+\frac{548289371}{453054264}v^2g_{c+\frac{7}{2}}-\frac{1453984}{1452097}v^2g_{c+\frac{15}{4}}+\frac{924056539}{3624434112}v^2q_{c+4}\\
y_{c+\frac{15}{4}}&=-\frac{224660513}{38660630528}y_c-\frac{863375625}{9665157632}y_{c+1}+\frac{41889472515}{38660630528}y_{c+2}+\frac{32325960975}{19330315264}vf_{c+3}-\frac{26287430631}{38660630528}v^2g_{c+3}+\frac{6514954677}{4832578816}v^2g_{c+\frac{7}{2}}-\frac{48336519}{46467104}v^2g_{c+\frac{15}{4}}+\frac{10208375793}{38660630528}v^2q_{c+4}\\
y_{c+4}&=\frac{111011}{18877261}y_c-\frac{1701312}{18877261}y_{c+1}+\frac{20467562}{18877261}y_{c+2}+\frac{36275232}{18877261}vf_{c+3}-\frac{178606424}{283158915}v^2g_{c+3}+\frac{422132864}{283158915}v^2g_{c+\frac{7}{2}}-\frac{22360064}{21781455}v^2g_{c+\frac{15}{4}}+\frac{78773252}{283158915}v^2q_{c+4}
\end{aligned} \tag{18}$$

D. Construction of HESDBBDF Method with Integrated One Off-grid Extended Future Point and One Extended Future Point for $k = 4$

With the same procedure, we integrated one off-grid extended future points at $x = x_{c+\frac{9}{2}}$ and one extended future point at $x = x_{c+5}$ as collocation points, thus the interpolation points, $j = 4$ and the collocation points $s = 4$ are considered, therefore, (5) becomes

$$y(x)=\alpha_0(x)y_c+\alpha_1(x)y_{c+1}+\alpha_2(x)y_{c+2}+\alpha_3(x)y_{c+3}+v[\beta_4(x)f_{c+4}+v\gamma_4(x)g_{c+4}+v\varepsilon_{\frac{9}{2}}(x)g_{c+\frac{9}{2}}+v\omega_5(x)q_{c+5}] \tag{19}$$

The matrix P in (10) becomes

$$P=\begin{pmatrix}
1 & x_c & x_c^2 & x_c^3 & x_c^4 & x_c^5 & x_c^6 & x_c^7 \\
1 & x_c+v & (x_c+v)^2 & (x_c+v)^3 & (x_c+v)^4 & (x_c+v)^5 & (x_c+v)^6 & (x_c+v)^7 \\
1 & x_c+2v & (x_c+2v)^2 & (x_c+2v)^3 & (x_c+2v)^4 & (x_c+2v)^5 & (x_c+2v)^6 & (x_c+2v)^7 \\
1 & x_c+3v & (x_c+3v)^2 & (x_c+3v)^3 & (x_c+3v)^4 & (x_c+3v)^5 & (x_c+3v)^6 & (x_c+3v)^7 \\
0 & 1 & 2x_c+8v & 3(x_c+4v)^2 & 4(x_c+4v)^3 & 5(x_c+4v)^4 & 6(x_c+4v)^5 & 7(x_c+4v)^6 \\
0 & 0 & 2 & 6x_c+24v & 12(x_c+4v)^2 & 20(x_c+4v)^3 & 30(x_c+4v)^4 & 42(x_c+4v)^5 \\
0 & 0 & 2 & 6x_c+27v & 12\left(x_c+\frac{9}{2}v\right)^2 & 20\left(x_c+\frac{9}{2}v\right)^3 & 30\left(x_c+\frac{9}{2}v\right)^4 & 42\left(x_c+\frac{9}{2}v\right)^5 \\
0 & 0 & 2 & 6x_c+30v & 12(x_c+5v)^2 & 20(x_c+5v)^3 & 30(x_c+5v)^4 & 42(x_c+5v)^5
\end{pmatrix} \tag{20}$$

The inverse of the matrix $N = P^{-1}$ is evaluated using Maple 18 from which the continuous scheme is obtained using (5), evaluating and simplifying it at $x = x_{c+4}, x = x_{c+\frac{9}{2}}, x = x_{c+5}$ and its derivative at $x = x_{c+1}, x = x_{c+2}$ and $x = x_{c+3}$ the following discrete schemes are obtained

$$\begin{aligned}
y_{c+1}&=-\frac{6611194}{8119095}v^2g_{c+4}+\frac{3940736}{8119095}v^2g_{c+\frac{9}{2}}-\frac{4442006}{40595475}v^2q_{c+5}-\frac{26663494}{40595475}vf_{c+1}+\frac{19370614}{40595475}vf_{c+4}-\frac{1184599}{13531825}y_c+\frac{28186487}{13531825}y_{c+2}-\frac{13470063}{13531825}y_{c+3}\\
y_{c+2}&=\frac{395872742}{590161785}v^2g_{c+4}-\frac{214095616}{590161785}v^2g_{c+\frac{9}{2}}+\frac{45955496}{590161785}v^2q_{c+5}-\frac{239971446}{196720595}vf_{c+2}+\frac{90831684}{196720595}vf_{c+4}+\frac{4928131}{196720595}y_c-\frac{74433464}{196720595}y_{c+1}+\frac{266225928}{196720595}y_{c+3}
\end{aligned}$$

$$y_{c+3}=\frac{43795628}{44681305}v^2g_{c+4}-\frac{19560448}{44681305}v^2g_{c+\frac{9}{2}}+\frac{3878126}{44681305}v^2q_{c+5}-\frac{79990482}{44681305}vf_{c+3}-\frac{41632092}{44681305}vf_{c+4}+\frac{662506}{44681305}y_c-\frac{7647927}{44681305}y_{c+1}+\frac{51666726}{44681305}y_{c+2}$$

$$y_{c+4}=-\frac{3369}{1025519}y_c-\frac{479168}{13331747}y_{c+1}-\frac{2795592}{13331747}y_{c+2}+\frac{15691968}{13331747}y_{c+3}+\frac{11363100}{13331747}vf_{c+4}-\frac{5552072}{13331747}v^2g_{c+4}+\frac{1828864}{13331747}v^2g_{c+\frac{9}{2}}-\frac{337472}{13331747}v^2q_{c+5}$$

$$y_{c+\frac{9}{2}}=-\frac{227185}{65633216}y_c+\frac{32121945}{853231808}y_{c+1}-\frac{185302467}{853231808}y_{c+2}+\frac{1009365735}{853231808}y_{c+3}-\frac{143741115}{106653976}vf_{c+4}-\frac{277116945}{853231808}v^2g_{c+4}+\frac{19198095}{106653976}v^2g_{c+\frac{9}{2}}-\frac{24813285}{853231808}v^2q_{c+5} \qquad (21)$$

$$y_{c+5}=-\frac{10918}{3076557}y_c+\frac{1543415}{39995241}y_{c+1}-\frac{8878930}{39995241}y_{c+2}+\frac{15824230}{13331747}y_{c+3}+\frac{24590860}{13331747}vf_{c+4}-\frac{25957700}{119985723}v^2g_{c+4}+\frac{52145920}{119985723}v^2g_{c+\frac{9}{2}}-\frac{1560500}{119985723}v^2q_{c+5}$$

III. CONVERGENCE ANALYSIS

Here, the examinations of order, error constant, consistency, zero stability and region of the absolute stability of (15), (18) and (21) are presented.

E. Order and Error Constant

The HESDBBDFM (5) is said to be of order u if $C_0 = C_1 = \ldots C_u = 0$ and the first non-zero coefficient $C_{u+1} \neq 0$ is the error constant as developed by [16] .The order and error constant for (15) are obtained as follows

$$C_0=\alpha_0+\alpha_1+\alpha_2+\alpha_{\frac{9}{4}}+\alpha_{\frac{5}{2}}+\alpha_{\frac{11}{4}}+\alpha_3=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

$$C_1=\alpha_1+2\alpha_2+\frac{9}{4}\alpha_{\frac{9}{4}}+\frac{5}{2}\alpha_{\frac{5}{2}}+\frac{11}{4}\alpha_{\frac{11}{4}}+3\alpha_3-\beta_0-\beta_1-\beta_2-\gamma_2-\gamma_{\frac{9}{4}}-\gamma_{\frac{5}{2}}-\gamma_{\frac{11}{4}}-\delta_3=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T \text{but}$$

$$C_2=\frac{1}{2}\alpha_1+2\alpha_2+\frac{81}{32}\alpha_{\frac{9}{4}}+\frac{25}{8}\alpha_{\frac{5}{2}}+\frac{121}{32}\alpha_{\frac{11}{4}}+\frac{9}{2}\alpha_3-\beta_1-2\beta_2-2\gamma_2-\frac{9}{4}\gamma_{\frac{9}{4}}-\frac{5}{2}\gamma_{\frac{5}{2}}-\frac{11}{4}\gamma_{\frac{11}{4}}-3\delta_3=\left(\frac{28141}{4088},-\frac{15938}{34273},-\frac{15223455}{35095552},-\frac{93225}{274184},-\frac{6448673}{35095552},\frac{1200}{34273}\right)^T$$

Therefore, (15) has order $u=1$ and error constants,

$$\left(\frac{28141}{4088},-\frac{15938}{34273},-\frac{15223455}{35095552},-\frac{93225}{274184},-\frac{6448673}{35095552},\frac{1200}{34273}\right)^T$$

With the same approach, (18) can be presented as

$$C_0=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_{\frac{7}{2}}+\alpha_{\frac{15}{4}}+\alpha_4=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

$$C_1=\alpha_1+2\alpha_2+3\alpha_3+\frac{7}{2}\alpha_{\frac{7}{2}}+\frac{15}{4}\alpha_{\frac{15}{4}}+4\alpha_4-\beta_0-\beta_1-\beta_2-\beta_3-\gamma_3-\gamma_{\frac{7}{2}}-\gamma_{\frac{15}{4}}-\delta_4=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

but

$$C_2=\frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3+\frac{49}{8}\alpha_{\frac{7}{2}}+\frac{225}{32}\alpha_{\frac{15}{4}}+8\alpha_4-\beta_1-2\beta_2-3\beta_3-3\gamma_3-\frac{7}{2}\gamma_{\frac{7}{2}}-\frac{15}{4}\gamma_{\frac{15}{4}}-4\delta_4=\left(\frac{24293087}{44821508},\frac{64547210}{36115559},-\frac{568098}{1452097},-\frac{6160385}{23233552},-\frac{160592355}{1486947328},\frac{162148}{1452097}\right)^T$$

Therefore, (18) has order $u=1$ and error constants,

$$\left(\frac{24293087}{44821508},\frac{64547210}{36115559},-\frac{568098}{1452097},-\frac{6160385}{23233552},-\frac{160592355}{1486947328},\frac{162148}{1452097}\right)^T$$

Applying the same approach, (21) can be obtained as

$$C_0=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_{\frac{9}{2}}+\alpha_5=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

$$C_1=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+\frac{9}{2}\alpha_{\frac{9}{2}}+5\alpha_5-\beta_0-\beta_1-\beta_2-\beta_3-\beta_4-\gamma_4-\gamma_{\frac{9}{2}}-\delta_5=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

but

$$C_2=\frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3+8\alpha_4+\frac{81}{8}\alpha_{\frac{9}{2}}+\frac{25}{2}\alpha_5-\beta_1-2\beta_2-3\beta_3-4\beta_4-4\gamma_4-\frac{9}{2}\gamma_{\frac{9}{2}}-5\delta_5=\left(-\frac{5931432}{13531825},\frac{75910874}{196720595},\frac{28113306}{44681305},-\frac{312360}{1025519},-\frac{5705595}{32816608},\frac{631480}{3076557}\right)^T$$

Therefore, (21) has order $u=1$ and error constants,

$$\left(-\frac{5931432}{13531825},\frac{75910874}{196720595},\frac{28113306}{44681305},-\frac{312360}{1025519},-\frac{5705595}{32816608},\frac{631480}{3076557}\right)^T$$

F. Consistency

The HESDBBDFM (5) is said to be consistent if the order u is greater than 1 such that the local truncation error (LTE) is $\|E_\psi\| = C_{u+1}v^{u+1} + O\left(v^{u+2}\right)$ where the maximum norm formulated by [16] is $\|\cdot\|$. The LTE of this method HESDBBDFM (5) is $C_{u+1} \neq 0$ as $u=1$.

From section 3.0 above, the local truncation error $C_{u+1} \neq 0$ for (15) is obtained and presented in vector representation as

$$C_2=\left(\frac{28141}{4088},-\frac{15938}{34273},-\frac{15223455}{35095552},-\frac{93225}{274184},-\frac{6448673}{35095552},\frac{1200}{34273}\right)^T$$

with order $u=(1,1,1,1,1,1)^T$.

Applying the same procedure for (18), the local truncation error $C_{u+1} \neq 0$ for (18) is obtained and presented in vector representation as

$$C_2=\left(\frac{24293087}{44821508},\frac{64547210}{36115559},-\frac{568098}{1452097},-\frac{6160385}{23233552},-\frac{160592355}{1486947328},\frac{162148}{1452097}\right)^T$$

with order $u=(1,1,1,1,1,1)^T$.

With the same procedure, (21), the local truncation error $C_{u+1} \neq 0$ for (21) is obtained and presented in vector representation as

$$C_2=\left(-\frac{5931432}{13531825},\frac{75910874}{196720595},\frac{28113306}{44681305},-\frac{312360}{1025519},-\frac{5705595}{32816608},\frac{631480}{3076557}\right)^T$$

with order $u=(1,1,1,1,1,1)^T$.

Since the local truncation error $C_{v+1} \neq 0$ of this method HESDBBDFM (5) tends to zero faster the step size v as v goes to zero and the order $u=1$ in (15), (18) and (21) satisfied the condition for consistency of order $u \geq 1$, then the proposed method is consistent.

G. Zero Stability

The zero stability for (15) is evaluated as follows

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0\\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0\\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0\\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0\\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}y_{c+1}\\ y_{c+2}\\ y_{c+\frac{9}{4}}\\ y_{c+\frac{5}{2}}\\ y_{c+\frac{11}{4}}\\ y_{c+3}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273}\\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776}\\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546}\\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776}\\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273}\end{pmatrix}\begin{pmatrix}y_{c-\frac{11}{4}}\\ y_{c-\frac{5}{2}}\\ y_{c-\frac{9}{4}}\\ y_{c-2}\\ y_{c-1}\\ y_c\end{pmatrix}$$

$$\begin{pmatrix}\frac{34273}{4088} & -\frac{30185}{4088} & 0 & 0 & 0 & 0\\ 0 & \frac{33474}{34273} & 0 & 0 & 0 & 0\\ 0 & \frac{21525525}{17547776} & 0 & 0 & 0 & 0\\ 0 & \frac{50610}{34273} & 0 & 0 & 0 & 0\\ 0 & \frac{30299115}{17547776} & 0 & 0 & 0 & 0\\ 0 & \frac{67746}{34273} & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}f_{c+1}\\ f_{c+2}\\ f_{c+\frac{9}{4}}\\ f_{c+\frac{5}{2}}\\ f_{c+\frac{11}{4}}\\ f_{c+3}\end{pmatrix}+v\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}f_{c-\frac{11}{4}}\\ f_{c-\frac{5}{2}}\\ f_{c-\frac{9}{4}}\\ f_{c-2}\\ f_{c-1}\\ f_c\end{pmatrix}$$

$$+v^2\begin{pmatrix}0 & \frac{3195957}{40880} & -\frac{4365586}{22995} & \frac{1204859}{6132} & -\frac{105674}{1095} & 0\\ 0 & -\frac{1591042}{514095} & \frac{3420416}{514095} & -\frac{3376096}{514095} & \frac{537344}{171365} & 0\\ 0 & -\frac{108033821}{35095552} & \frac{14640587}{2193472} & -\frac{115404915}{17547776} & \frac{6885593}{2193472} & 0\\ 0 & -\frac{3352191}{1096736} & \frac{1387699}{205638} & -\frac{3609059}{548368} & \frac{215441}{68546} & 0\\ 0 & -\frac{1597733137}{526433280} & \frac{224660513}{32902080} & -\frac{1720324991}{263216640} & \frac{34581701}{10967360} & 0\\ 0 & -\frac{309899}{102819} & \frac{3552352}{514095} & -\frac{1111732}{171365} & \frac{1653856}{514095} & 0\end{pmatrix}\begin{pmatrix}g_{c+1}\\ g_{c+2}\\ g_{c+\frac{9}{4}}\\ g_{c+\frac{5}{2}}\\ g_{c+\frac{11}{4}}\\ g_{c+3}\end{pmatrix}+v^2\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}g_{c-\frac{11}{4}}\\ g_{c-\frac{5}{2}}\\ g_{c-\frac{9}{4}}\\ g_{c-2}\\ g_{c-1}\\ g_c\end{pmatrix}$$

$$+v^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6833377}{367920} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20292}{34273} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5199671}{8773888} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1952125}{3290208} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3258717}{5483680} \\ 0 & 0 & 0 & 0 & 0 & -\frac{303517}{514095} \end{pmatrix}\begin{pmatrix} q_{c+1} \\ q_{c+2} \\ q_{c+\frac{9}{4}} \\ q_{c+\frac{5}{2}} \\ q_{c+\frac{11}{4}} \\ q_{c+3} \end{pmatrix}+v^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_{c-\frac{11}{4}} \\ q_{c-\frac{5}{2}} \\ q_{c-\frac{9}{4}} \\ q_{c-2} \\ q_{c-1} \\ q_{c} \end{pmatrix}$$

where

$$Z_2^{(1)}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0 \\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0 \\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0 \\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Z_1^{(1)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix},$$

$$U_2^{(1)}=\begin{pmatrix} \frac{34273}{4088} & -\frac{30185}{4088} & 0 & 0 & 0 & 0 \\ 0 & \frac{33474}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{21525525}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{50610}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{30299115}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{67746}{34273} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_2^{(1)}=\begin{pmatrix} 0 & \frac{3195957}{40880} & -\frac{4365586}{22995} & \frac{1204859}{6132} & -\frac{105674}{1095} & 0 \\ 0 & -\frac{1591042}{514095} & \frac{3420416}{514095} & -\frac{3376096}{514095} & \frac{537344}{171365} & 0 \\ 0 & -\frac{108033821}{35095552} & \frac{14640587}{2193472} & -\frac{115404915}{17547776} & \frac{6885593}{2193472} & 0 \\ 0 & -\frac{3352191}{1096736} & \frac{1387699}{205638} & -\frac{3609059}{548368} & \frac{215441}{68546} & 0 \\ 0 & -\frac{1597733137}{526433280} & \frac{224660513}{32902080} & -\frac{1720324991}{263216640} & \frac{34581701}{10967360} & 0 \\ 0 & -\frac{309899}{102819} & \frac{3552352}{514095} & -\frac{1111732}{171365} & \frac{1653856}{514095} & 0 \end{pmatrix} \text{and}$$

$$T_2^{(1)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6833377}{367920} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20292}{34273} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5199671}{8773888} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1952125}{3290208} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3258717}{5483680} \\ 0 & 0 & 0 & 0 & 0 & -\frac{303517}{514095} \end{pmatrix}$$

The first characteristic polynomial is stated as

$$\begin{aligned} R(a) &= \det\left(aZ_2^{(1)}-Z_1^{(1)}\right) \\ &= \left|aZ_2^{(1)}-Z_1^{(1)}\right| = 0 \end{aligned} \tag{22}$$

Now we have,

$$R(a)=\left|a\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0 \\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0 \\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0 \\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273}a & a & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776}a & 0 & a & 0 & 0 & 0 \\ -\frac{70145}{68546}a & 0 & 0 & a & 0 & 0 \\ -\frac{17957269}{17547776}a & 0 & 0 & 0 & a & 0 \\ -\frac{35073}{34273}a & 0 & 0 & 0 & 0 & a \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix}\right|$$

$$\Rightarrow R(a)=\begin{vmatrix} a & 0 & 0 & 0 & 0 & 1 \\ -\frac{35072}{34273}a & a & 0 & 0 & 0 & -\frac{799}{34273} \\ -\frac{17956971}{17547776}a & 0 & a & 0 & 0 & -\frac{409195}{17547776} \\ -\frac{70145}{68546}a & 0 & 0 & a & 0 & -\frac{1599}{68546} \\ -\frac{17957269}{17547776}a & 0 & 0 & 0 & a & -\frac{409493}{17547776} \\ -\frac{35073}{34273}a & 0 & 0 & 0 & 0 & a-\frac{800}{34273} \end{vmatrix}$$

Using Maple (18) software, we obtain

$$R(a)=a^5(a+1)$$

$$\Rightarrow a^5(a+1)=0$$

$\Rightarrow a_1=-1, a_2=0, a_3=0, a_4=0, a_5=0, a_6=0$. Since $|a_i|<1,\ i=1,2,3,4,5,6$, (15) is zero stable.

Applying the same approach, then **(18)** is presented as

$$\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0 \\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0 \\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0 \\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{c+1} \\ y_{c+2} \\ y_{c+3} \\ y_{c+\frac{7}{2}} \\ y_{c+\frac{15}{4}} \\ y_{c+4} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix}\begin{pmatrix} y_{c-\frac{15}{4}} \\ y_{c-\frac{7}{2}} \\ y_{c-3} \\ y_{c-2} \\ y_{c-1} \\ y_{c} \end{pmatrix}$$

$$+v\begin{pmatrix} -\frac{56631783}{89643016} & 0 & -\frac{46703841}{89643016} & 0 & 0 & 0 \\ 0 & \frac{113263566}{36115559} & -\frac{79330800}{36115559} & 0 & 0 & 0 \\ 0 & 0 & \frac{17437098}{18877261} & 0 & 0 & 0 \\ 0 & 0 & \frac{859557615}{604072352} & 0 & 0 & 0 \\ 0 & 0 & \frac{32325960975}{19330315264} & 0 & 0 & 0 \\ 0 & 0 & \frac{36275232}{18877261} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{c+1} \\ f_{c+2} \\ f_{c+3} \\ f_{c+\frac{7}{2}} \\ f_{c+\frac{15}{4}} \\ f_{c+4} \end{pmatrix}+v\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{c-\frac{15}{4}} \\ f_{c-\frac{7}{2}} \\ f_{c-3} \\ f_{c-2} \\ f_{c-1} \\ f_{c} \end{pmatrix}$$

$$+v^2\begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0 \\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0 \\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0 \\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0 \\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0 \\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0 \end{pmatrix}\begin{pmatrix} g_{c+1} \\ g_{c+2} \\ g_{c+3} \\ g_{c+\frac{7}{2}} \\ g_{c+\frac{15}{4}} \\ g_{c+4} \end{pmatrix}+v^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{c-\frac{15}{4}} \\ g_{c-\frac{7}{2}} \\ g_{c-3} \\ g_{c-2} \\ g_{c-1} \\ g_{c} \end{pmatrix}$$

$$+v^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385} \\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305} \\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112} \\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915} \end{pmatrix}\begin{pmatrix} q_{c+1} \\ q_{c+2} \\ q_{c+3} \\ q_{c+\frac{7}{2}} \\ q_{c+\frac{15}{4}} \\ q_{c+4} \end{pmatrix}+v^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_{c-\frac{15}{4}} \\ q_{c-\frac{7}{2}} \\ q_{c-3} \\ q_{c-2} \\ q_{c-1} \\ q_{c} \end{pmatrix}$$

where

$$Z_2^{(2)}=\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0 \\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0 \\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0 \\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1 \end{pmatrix}, Z_1^{(2)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix},$$

$$U_2^{(2)}=\begin{pmatrix} -\frac{56631783}{89643016} & 0 & -\frac{46703841}{89643016} & 0 & 0 & 0 \\ 0 & \frac{113263566}{36115559} & -\frac{79330800}{36115559} & 0 & 0 & 0 \\ 0 & 0 & \frac{17437098}{18877261} & 0 & 0 & 0 \\ 0 & 0 & \frac{859557615}{604072352} & 0 & 0 & 0 \\ 0 & 0 & \frac{32325960975}{19330315264} & 0 & 0 & 0 \\ 0 & 0 & \frac{36275232}{18877261} & 0 & 0 & 0 \end{pmatrix}$$

$$H_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0 \\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0 \\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0 \\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0 \\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0 \\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0 \end{pmatrix} \text{and}$$

$$T_2^{(2)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385} \\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305} \\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112} \\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915} \end{pmatrix}$$

The first characteristic polynomial is presented as

$$R(a)=\det\left(aZ_2^{(2)}-Z_1^{(2)}\right) = \left|aZ_2^{(2)}-Z_1^{(2)}\right|=0 \qquad (23)$$

Now we have,

$$R(a)=\left| a\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0 \\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0 \\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0 \\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} a & -\frac{12061165}{11205377}a & 0 & 0 & 0 & 0 \\ -\frac{38298352}{36115559}a & a & 0 & 0 & 0 & 0 \\ \frac{1653939}{18877261}a & \frac{20424312}{18877261}a & a & 0 & 0 & 0 \\ \frac{1671544}{18877261}a & -\frac{1308185025}{1208144704}a & 0 & a & 0 & 0 \\ \frac{863375625}{9665157632}a & -\frac{41889472515}{38660630528}a & 0 & 0 & a & 0 \\ \frac{1701312}{18877261}a & -\frac{20467562}{18877261}a & 0 & 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377} \\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559} \\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704} \\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261} \end{pmatrix} \right|$$

$$\Rightarrow R(a)=\begin{pmatrix} a & -\frac{12061165}{11205377}a & 0 & 0 & 0 & -\frac{855788}{11205377} \\ -\frac{38298352}{36115559}a & a & 0 & 0 & 0 & -\frac{2182793}{36115559} \\ \frac{1653939}{18877261}a & \frac{20424312}{18877261}a & a & 0 & 0 & \frac{106888}{18877261} \\ \frac{1671544}{18877261}a & -\frac{1308185025}{1208144704}a & 0 & a & 0 & \frac{6938495}{1208144704} \\ \frac{863375625}{9665157632}a & -\frac{41889472515}{38660630528}a & 0 & 0 & a & \frac{224660513}{38660630528} \\ \frac{1701312}{18877261}a & -\frac{20467562}{18877261}a & 0 & 0 & 0 & a+\frac{111011}{18877261} \end{pmatrix}$$

Using Maple (18) software, we obtain

$$R(a)=-\frac{57234288539337}{404688454160743}a^5(a+1)$$

$$\Rightarrow -\frac{57234288539337}{404688454160743}a^5(a+1)=0$$

$\Rightarrow a_1=-1, a_2=0, a_3=0, a_4=0, a_5=0, a_6=0$. Since $|a_i|<1,\ i=1,2,3,4,5,6$, (18) is zero stable.

With the same procedure, (21) can be presented as follows

$$\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{c+1} \\ y_{c+2} \\ y_{c+3} \\ y_{c+4} \\ y_{c+\frac{9}{2}} \\ y_{c+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595} \\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305} \\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519} \\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216} \\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557} \end{pmatrix}\begin{pmatrix} y_{c-\frac{9}{2}} \\ y_{c-4} \\ y_{c-3} \\ y_{c-2} \\ y_{c-1} \\ y_c \end{pmatrix}$$

$$+v\begin{pmatrix} -\frac{26663494}{40595475} & 0 & 0 & \frac{19370614}{40595475} & 0 & 0 \\ 0 & -\frac{239971446}{196720595} & 0 & -\frac{90831684}{196720595} & 0 & 0 \\ 0 & 0 & \frac{79990482}{44681305} & -\frac{41632092}{44681305} & 0 & 0 \\ 0 & 0 & 0 & \frac{11363100}{13331747} & 0 & 0 \\ 0 & 0 & 0 & \frac{143741115}{106653976} & 0 & 0 \\ 0 & 0 & 0 & \frac{24590860}{13331747} & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{c+1} \\ f_{c+2} \\ f_{c+3} \\ f_{c+4} \\ f_{c+\frac{9}{2}} \\ f_{c+5} \end{pmatrix}+v\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} f_{c-\frac{9}{2}} \\ f_{c-4} \\ f_{c-3} \\ f_{c-2} \\ f_{c-1} \\ f_c \end{pmatrix}$$

$$+v^2\begin{pmatrix} 0 & 0 & 0 & -\frac{6611194}{8119095} & \frac{3940736}{8119095} & 0 \\ 0 & 0 & 0 & \frac{395872742}{590161785} & -\frac{214095616}{590161785} & 0 \\ 0 & 0 & 0 & \frac{43795628}{44681305} & -\frac{19560448}{44681305} & 0 \\ 0 & 0 & 0 & -\frac{5552072}{13331747} & \frac{1828864}{13331747} & 0 \\ 0 & 0 & 0 & -\frac{277116945}{853231808} & \frac{19198095}{106653976} & 0 \\ 0 & 0 & 0 & \frac{25957700}{119985723} & \frac{52145920}{119985723} & 0 \end{pmatrix}\begin{pmatrix} g_{c+1} \\ g_{c+2} \\ g_{c+3} \\ g_{c+4} \\ g_{c+\frac{9}{2}} \\ g_{c+5} \end{pmatrix}+v^2\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} g_{c-\frac{9}{2}} \\ g_{c-4} \\ g_{c-3} \\ g_{c-2} \\ g_{c-1} \\ g_c \end{pmatrix}$$

$$+v^2\begin{pmatrix} 0&0&0&0&0& -\frac{4442006}{40595475} \\ 0&0&0&0&0& \frac{45955496}{590161785} \\ 0&0&0&0&0& \frac{3878126}{44681305} \\ 0&0&0&0&0& -\frac{337472}{13331747} \\ 0&0&0&0&0& -\frac{24813285}{853231808} \\ 0&0&0&0&0& -\frac{1560500}{119985723} \end{pmatrix}\begin{pmatrix} q_{c+1} \\ q_{c+2} \\ q_{c+3} \\ q_{c+4} \\ q_{c+\frac{9}{2}} \\ q_{c+5} \end{pmatrix}+v^2\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} q_{c-\frac{9}{2}} \\ q_{c-4} \\ q_{c-3} \\ q_{c-2} \\ q_{c-1} \\ q_c \end{pmatrix}\text{where}$$

$$Z_2^{(3)}=\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix}, Z_1^{(3)}=\begin{pmatrix} 0&0&0&0&0& \frac{1184599}{13531825} \\ 0&0&0&0&0& -\frac{4928131}{196720595} \\ 0&0&0&0&0& -\frac{662506}{44681305} \\ 0&0&0&0&0& \frac{3369}{1025519} \\ 0&0&0&0&0& \frac{227185}{65633216} \\ 0&0&0&0&0& \frac{10918}{3076557} \end{pmatrix},$$

$$U_2^{(3)}=\begin{pmatrix} -\frac{26663494}{40595475} & 0 & 0 & \frac{19370614}{40595475} & 0 & 0 \\ 0 & -\frac{239971446}{196720595} & 0 & -\frac{90831684}{196720595} & 0 & 0 \\ 0 & 0 & \frac{79990482}{44681305} & -\frac{41632092}{44681305} & 0 & 0 \\ 0 & 0 & 0 & \frac{11363100}{13331747} & 0 & 0 \\ 0 & 0 & 0 & \frac{143741115}{106653976} & 0 & 0 \\ 0 & 0 & 0 & \frac{24590860}{13331747} & 0 & 0 \end{pmatrix}, H_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{6611194}{8119095} & \frac{3940736}{8119095} & 0 \\ 0 & 0 & 0 & \frac{395872742}{590161785} & -\frac{214095616}{590161785} & 0 \\ 0 & 0 & 0 & \frac{43795628}{44681305} & -\frac{19560448}{44681305} & 0 \\ 0 & 0 & 0 & -\frac{5552072}{13331747} & \frac{1828864}{13331747} & 0 \\ 0 & 0 & 0 & -\frac{277116945}{853231808} & \frac{19198095}{106653976} & 0 \\ 0 & 0 & 0 & \frac{25957700}{119985723} & \frac{52145920}{119985723} & 0 \end{pmatrix},$$

and

$$T_2^{(3)}=\begin{pmatrix} 0&0&0&0&0& -\frac{4442006}{40595475} \\ 0&0&0&0&0& \frac{45955496}{590161785} \\ 0&0&0&0&0& \frac{3878126}{44681305} \\ 0&0&0&0&0& -\frac{337472}{13331747} \\ 0&0&0&0&0& -\frac{24813285}{853231808} \\ 0&0&0&0&0& -\frac{1560500}{119985723} \end{pmatrix}$$

Let the first characteristic polynomial be stated as

$$\begin{aligned} R(a) &= \det\left(aZ_2^{(3)}-Z_1^{(3)}\right) \\ &= \left|aZ_2^{(3)}-Z_1^{(3)}\right| = 0 \end{aligned} \tag{24}$$

Now we have,

$$R(a)=\left|a\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0 \\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0 \\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0 \\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0 \\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0 \\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0&0&0&0&0& \frac{1184599}{13531825} \\ 0&0&0&0&0& -\frac{4928131}{196720595} \\ 0&0&0&0&0& -\frac{662506}{44681305} \\ 0&0&0&0&0& \frac{3369}{1025519} \\ 0&0&0&0&0& \frac{227185}{65633216} \\ 0&0&0&0&0& \frac{10918}{3076557} \end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} a & -\frac{28186487}{13531825}a & \frac{13470063}{13531825}a & 0 & 0 & 0 \\ \frac{74433464}{196720595}a & a & -\frac{266225928}{196720595}a & 0 & 0 & 0 \\ \frac{7647927}{44681305}a & -\frac{51666726}{44681305}a & a & 0 & 0 & 0 \\ -\frac{479168}{13331747}a & \frac{2795592}{13331747}a & -\frac{15691968}{13331747}a & a & 0 & 0 \\ -\frac{32121945}{853231808}a & \frac{185302467}{853231808}a & -\frac{1009365735}{853231808}a & 0 & a & 0 \\ -\frac{1543415}{39995241}a & \frac{8878930}{39995241}a & -\frac{15824230}{13331747}a & 0 & 0 & a \end{pmatrix}-\begin{pmatrix} 0&0&0&0&0& \frac{1184599}{13531825} \\ 0&0&0&0&0& -\frac{4928131}{196720595} \\ 0&0&0&0&0& -\frac{662506}{44681305} \\ 0&0&0&0&0& \frac{3369}{1025519} \\ 0&0&0&0&0& \frac{227185}{65633216} \\ 0&0&0&0&0& \frac{10918}{3076557} \end{pmatrix}\right|$$

$$\Rightarrow R(a)=\begin{pmatrix} a & -\frac{28186487}{13531825}a & \frac{13470063}{13531825}a & 0 & 0 & -\frac{1184599}{13531825} \\ \frac{74433464}{196720595}a & a & -\frac{266225928}{196720595}a & 0 & 0 & \frac{4928131}{196720595} \\ \frac{7647927}{44681305}a & -\frac{51666726}{44681305}a & a & 0 & 0 & \frac{662506}{44681305} \\ -\frac{479168}{13331747}a & \frac{2795592}{13331747}a & -\frac{15691968}{13331747}a & a & 0 & -\frac{3369}{1025519} \\ -\frac{32121945}{853231808}a & \frac{185302467}{853231808}a & -\frac{1009365735}{853231808}a & 0 & a & -\frac{227185}{65633216} \\ -\frac{1543415}{39995241}a & \frac{8878930}{39995241}a & -\frac{15824230}{13331747}a & 0 & 0 & a-\frac{10918}{3076557} \end{pmatrix}$$

Using Maple (18) software, we obtain

$$R(a)=\frac{237490998930113171443\,2}{237882254933766577633\,75}a^5(a+1)$$

$$\Rightarrow \frac{2374909989301131714432}{2378822549337665776337 5}a^5(a+1)=0$$

$\Rightarrow a_1=-1, a_2=0, a_3=0, a_4=0, a_5=0, a_6=0$. Since $|a_i|<1,\ i=1,2,3,4,5,6$, (21) is zero stable.

H. Convergence

Since (15), (18) and (21) are both consistent and zero stable as proved in section 3.3 which satisfied the necessary and sufficient condition for convergence of a numerical method, therefore the proposed method is convergent.

I. Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the D- and E-stability by applying (15), (18) and (21) to the DDEs of this form:

$$\begin{aligned} &x'(t)=\eta x(t)+\varphi x(t-\tau), t\ge t_0 \\ &x(t)=\varsigma(t), \qquad t\le t_0 \end{aligned} \tag{25}$$

where $\varsigma(t)$ is the initial function, η,φ are complex coefficients, $\tau=cv, c\in Z^+$, v is the step size and $c=\frac{\tau}{v}$, c is a positive integer. Let $P_1=v\rho$ and $P_2=v\delta$, then from (15), we have

$$Y_{C+6}=\begin{pmatrix} y_{c+1} \\ y_{c+2} \\ y_{c+\frac{9}{4}} \\ y_{c+\frac{5}{2}} \\ y_{c+\frac{11}{4}} \\ y_{c+3} \end{pmatrix}, Y_C=\begin{pmatrix} y_{c-\frac{11}{4}} \\ y_{c-\frac{5}{2}} \\ y_{c-\frac{9}{4}} \\ y_{c-2} \\ y_{c-1} \\ y_c \end{pmatrix}, F_{C+6}=\begin{pmatrix} f_{c+1} \\ f_{c+2} \\ f_{c+\frac{9}{4}} \\ f_{c+\frac{5}{2}} \\ f_{c+\frac{11}{4}} \\ f_{c+3} \end{pmatrix}, F_C=\begin{pmatrix} f_{c-\frac{11}{4}} \\ f_{c-\frac{5}{2}} \\ f_{c-\frac{9}{4}} \\ f_{c-2} \\ f_{c-1} \\ f_c \end{pmatrix}, G_{C+6}=\begin{pmatrix} g_{c+1} \\ g_{c+2} \\ g_{c+\frac{9}{4}} \\ g_{c+\frac{5}{2}} \\ g_{c+\frac{11}{4}} \\ g_{c+3} \end{pmatrix}, G_C=\begin{pmatrix} g_{c-\frac{11}{4}} \\ g_{c-\frac{5}{2}} \\ g_{c-\frac{9}{4}} \\ g_{c-2} \\ g_{c-1} \\ g_c \end{pmatrix}, Q_{C+6}=\begin{pmatrix} q_{c+1} \\ q_{c+2} \\ q_{c+\frac{9}{4}} \\ q_{c+\frac{5}{2}} \\ q_{c+\frac{11}{4}} \\ q_{c+3} \end{pmatrix}$$ and

$$Q_C=\begin{pmatrix} q_{c-\frac{11}{4}} \\ q_{c-\frac{5}{2}} \\ q_{c-\frac{9}{4}} \\ q_{c-2} \\ q_{c-1} \\ q_c \end{pmatrix}$$. Since

$$Z_2^{(1)}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{35072}{34273} & 1 & 0 & 0 & 0 & 0 \\ -\frac{17956971}{17547776} & 0 & 1 & 0 & 0 & 0 \\ -\frac{70145}{68546} & 0 & 0 & 1 & 0 & 0 \\ -\frac{17957269}{17547776} & 0 & 0 & 0 & 1 & 0 \\ -\frac{35073}{34273} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Z_1^{(1)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{799}{34273} \\ 0 & 0 & 0 & 0 & 0 & \frac{409195}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{1599}{68546} \\ 0 & 0 & 0 & 0 & 0 & \frac{409493}{17547776} \\ 0 & 0 & 0 & 0 & 0 & \frac{800}{34273} \end{pmatrix},$$

$$U_2^{(1)}=\begin{pmatrix} \frac{34273}{4088} & -\frac{30185}{4088} & 0 & 0 & 0 & 0 \\ 0 & \frac{33474}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{21525525}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{50610}{34273} & 0 & 0 & 0 & 0 \\ 0 & \frac{30299115}{17547776} & 0 & 0 & 0 & 0 \\ 0 & \frac{67746}{34273} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_2^{(1)}=\begin{pmatrix} 0 & \frac{3195957}{40880} & -\frac{4365586}{22995} & \frac{1204859}{6132} & -\frac{105674}{1095} & 0 \\ 0 & -\frac{1591042}{514095} & \frac{3420416}{514095} & -\frac{3376096}{514095} & \frac{537344}{171365} & 0 \\ 0 & -\frac{108033821}{35095552} & \frac{14640587}{2193472} & -\frac{115404915}{17547776} & \frac{6885593}{2193472} & 0 \\ 0 & -\frac{3352191}{1096736} & \frac{1387699}{205638} & -\frac{3609059}{548368} & \frac{215441}{68546} & 0 \\ 0 & -\frac{1597733137}{526433280} & \frac{224660513}{32902080} & -\frac{1720324991}{263216640} & \frac{34581701}{10967360} & 0 \\ 0 & -\frac{309899}{102819} & \frac{3552352}{514095} & -\frac{1111732}{171365} & \frac{1653856}{514095} & 0 \end{pmatrix}$$ and

$$T_2^{(1)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6833377}{367920} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20292}{34273} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5199671}{8773888} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1952125}{3290208} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3258717}{5483680} \\ 0 & 0 & 0 & 0 & 0 & -\frac{303517}{514095} \end{pmatrix}$$

we have,

$$Z_2^{(1)}Y_{C+2}=Z_1^{(1)}Y_{C+1}-v\sum_{i=1}^{2}\left(U_i^{(1)}F_{C+i}+vH_i^{(1)}G_{C+i}+vT_i^{(1)}Q_{C+i}\right) \tag{26}$$

With the same technique for (18), we have

$$Y_{C+6}=\begin{pmatrix} y_{c+1}\\ y_{c+2}\\ y_{c+3}\\ y_{c+\frac{7}{2}}\\ y_{c+\frac{15}{4}}\\ y_{c+4}\end{pmatrix},Y_C=\begin{pmatrix} y_{c-\frac{15}{4}}\\ y_{c-\frac{7}{2}}\\ y_{c-3}\\ y_{c-2}\\ y_{c-1}\\ y_{c}\end{pmatrix},F_{C+6}=\begin{pmatrix} f_{c+1}\\ f_{c+2}\\ f_{c+3}\\ f_{c+\frac{7}{2}}\\ f_{c+\frac{15}{4}}\\ f_{c+4}\end{pmatrix},F_C=\begin{pmatrix} f_{c-\frac{15}{4}}\\ f_{c-\frac{7}{2}}\\ f_{c-3}\\ f_{c-2}\\ f_{c-1}\\ f_{c}\end{pmatrix},G_{C+6}=\begin{pmatrix} g_{c+1}\\ g_{c+2}\\ g_{c+3}\\ g_{c+\frac{7}{2}}\\ g_{c+\frac{15}{4}}\\ g_{c+4}\end{pmatrix},G_C=\begin{pmatrix} g_{c-\frac{15}{4}}\\ g_{c-\frac{7}{2}}\\ g_{c-3}\\ g_{c-2}\\ g_{c-1}\\ g_{c}\end{pmatrix},Q_{C+6}=\begin{pmatrix} q_{c+1}\\ q_{c+2}\\ q_{c+3}\\ q_{c+\frac{7}{2}}\\ q_{c+\frac{15}{4}}\\ q_{c+4}\end{pmatrix}$$ and

$$Q_C=\begin{pmatrix} q_{c-\frac{15}{4}}\\ q_{c-\frac{7}{2}}\\ q_{c-3}\\ q_{c-2}\\ q_{c-1}\\ q_{c}\end{pmatrix}$$ .Since

$$Z_2^{(2)}=\begin{pmatrix} 1 & -\frac{12061165}{11205377} & 0 & 0 & 0 & 0\\ -\frac{38298352}{36115559} & 1 & 0 & 0 & 0 & 0\\ \frac{1653939}{18877261} & \frac{20424312}{18877261} & 1 & 0 & 0 & 0\\ \frac{1671544}{18877261} & -\frac{1308185025}{1208144704} & 0 & 1 & 0 & 0\\ \frac{863375625}{9665157632} & -\frac{41889472515}{38660630528} & 0 & 0 & 1 & 0\\ \frac{1701312}{18877261} & -\frac{20467562}{18877261} & 0 & 0 & 0 & 1\end{pmatrix},Z_1^{(2)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{855788}{11205377}\\ 0 & 0 & 0 & 0 & 0 & \frac{2182793}{36115559}\\ 0 & 0 & 0 & 0 & 0 & -\frac{106888}{18877261}\\ 0 & 0 & 0 & 0 & 0 & -\frac{6938495}{1208144704}\\ 0 & 0 & 0 & 0 & 0 & -\frac{224660513}{38660630528}\\ 0 & 0 & 0 & 0 & 0 & -\frac{111011}{18877261}\end{pmatrix},$$

$$U_2^{(2)}=\begin{pmatrix} -\frac{56631783}{89643016} & 0 & -\frac{46703841}{89643016} & 0 & 0 & 0\\ 0 & \frac{113263566}{36115559} & -\frac{79330800}{36115559} & 0 & 0 & 0\\ 0 & 0 & \frac{17437098}{18877261} & 0 & 0 & 0\\ 0 & 0 & \frac{859557615}{604072352} & 0 & 0 & 0\\ 0 & 0 & \frac{32325960975}{19330315264} & 0 & 0 & 0\\ 0 & 0 & \frac{36275232}{18877261} & 0 & 0 & 0\end{pmatrix}$$

$$H_2^{(2)}=\begin{pmatrix} 0 & 0 & \frac{626087947}{336161310} & -\frac{680587916}{168080655} & \frac{630800768}{168080655} & 0\\ 0 & 0 & \frac{2573430208}{541733385} & -\frac{4603388288}{541733385} & \frac{1356812288}{180577795} & 0\\ 0 & 0 & -\frac{75518022}{94386305} & \frac{105233472}{94386305} & -\frac{6899712}{7260485} & 0\\ 0 & 0 & -\frac{2642247503}{3624434112} & \frac{548289371}{453054264} & -\frac{1453984}{1452097} & 0\\ 0 & 0 & -\frac{26287430631}{38660630528} & \frac{6514954677}{4832578816} & -\frac{48336519}{46467104} & 0\\ 0 & 0 & -\frac{178606424}{283158915} & \frac{422132864}{283158915} & -\frac{22360064}{21781455} & 0\end{pmatrix}$$

and $$T_2^{(2)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{688630997}{672322620}\\ 0 & 0 & 0 & 0 & 0 & -\frac{1072270634}{541733385}\\ 0 & 0 & 0 & 0 & 0 & \frac{23054436}{94386305}\\ 0 & 0 & 0 & 0 & 0 & \frac{924056539}{3624434112}\\ 0 & 0 & 0 & 0 & 0 & \frac{10208375793}{38660630528}\\ 0 & 0 & 0 & 0 & 0 & \frac{78773252}{283158915}\end{pmatrix}$$

we have,

$$Z_2^{(2)}Y_{C+2}=Z_1^{(2)}Y_{C+1}-v\sum_{i=1}^{2}\left(U_i^{(2)}F_{C+i}+vH_i^{(2)}G_{C+i}+vT_i^{(2)}Q_{C+i}\right)(27)$$

Applying the same approach for (21), we have

$$Y_{C+6}=\begin{pmatrix} y_{c+1}\\ y_{c+2}\\ y_{c+3}\\ y_{c+4}\\ y_{c+\frac{9}{2}}\\ y_{c+5}\end{pmatrix},Y_C=\begin{pmatrix} y_{c-\frac{9}{2}}\\ y_{c-4}\\ y_{c-3}\\ y_{c-2}\\ y_{c-1}\\ y_{c}\end{pmatrix},F_{C+6}=\begin{pmatrix} f_{c+1}\\ f_{c+2}\\ f_{c+3}\\ f_{c+4}\\ f_{c+\frac{9}{2}}\\ f_{c+5}\end{pmatrix},F_C=\begin{pmatrix} f_{c-\frac{9}{2}}\\ f_{c-4}\\ f_{c-3}\\ f_{c-2}\\ f_{c-1}\\ f_{c}\end{pmatrix},G_{C+6}=\begin{pmatrix} g_{c+1}\\ g_{c+2}\\ g_{c+3}\\ g_{c+4}\\ g_{c+\frac{9}{2}}\\ g_{c+5}\end{pmatrix},G_C=\begin{pmatrix} g_{c-\frac{9}{2}}\\ g_{c-4}\\ g_{c-3}\\ g_{c-2}\\ g_{c-1}\\ g_{c}\end{pmatrix},Q_{C+6}=\begin{pmatrix} q_{c+1}\\ q_{c+2}\\ q_{c+3}\\ q_{c+4}\\ q_{c+\frac{9}{2}}\\ q_{c+5}\end{pmatrix}$$ and

$$Q_C=\begin{pmatrix} q_{c-\frac{9}{2}}\\ q_{c-4}\\ q_{c-3}\\ q_{c-2}\\ q_{c-1}\\ q_{c}\end{pmatrix}$$ .Since

$$Z_2^{(3)}=\begin{pmatrix} 1 & -\frac{28186487}{13531825} & \frac{13470063}{13531825} & 0 & 0 & 0\\ \frac{74433464}{196720595} & 1 & -\frac{266225928}{196720595} & 0 & 0 & 0\\ \frac{7647927}{44681305} & -\frac{51666726}{44681305} & 1 & 0 & 0 & 0\\ -\frac{479168}{13331747} & \frac{2795592}{13331747} & -\frac{15691968}{13331747} & 1 & 0 & 0\\ -\frac{32121945}{853231808} & \frac{185302467}{853231808} & -\frac{1009365735}{853231808} & 0 & 1 & 0\\ -\frac{1543415}{39995241} & \frac{8878930}{39995241} & -\frac{15824230}{13331747} & 0 & 0 & 1\end{pmatrix},Z_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1184599}{13531825}\\ 0 & 0 & 0 & 0 & 0 & -\frac{4928131}{196720595}\\ 0 & 0 & 0 & 0 & 0 & -\frac{662506}{44681305}\\ 0 & 0 & 0 & 0 & 0 & \frac{3369}{1025519}\\ 0 & 0 & 0 & 0 & 0 & \frac{227185}{65633216}\\ 0 & 0 & 0 & 0 & 0 & \frac{10918}{3076557}\end{pmatrix},$$

$$U_2^{(3)}=\begin{pmatrix} -\frac{26663494}{40595475} & 0 & 0 & \frac{19370614}{40595475} & 0 & 0 \\ 0 & -\frac{239971446}{196720595} & 0 & -\frac{90831684}{196720595} & 0 & 0 \\ 0 & 0 & \frac{79990482}{44681305} & -\frac{41632092}{44681305} & 0 & 0 \\ 0 & 0 & 0 & \frac{11363100}{13331747} & 0 & 0 \\ 0 & 0 & 0 & \frac{143741115}{106653976} & 0 & 0 \\ 0 & 0 & 0 & \frac{24590860}{13331747} & 0 & 0 \end{pmatrix}, H_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & -\frac{6611194}{8119095} & \frac{3940736}{8119095} & 0 \\ 0 & 0 & 0 & \frac{395872742}{590161785} & -\frac{214095616}{590161785} & 0 \\ 0 & 0 & 0 & \frac{43795628}{44681305} & -\frac{19560448}{44681305} & 0 \\ 0 & 0 & 0 & -\frac{5552072}{13331747} & \frac{1828864}{13331747} & 0 \\ 0 & 0 & 0 & -\frac{277116945}{853231808} & \frac{19198095}{106653976} & 0 \\ 0 & 0 & 0 & \frac{25957700}{119985723} & \frac{52145920}{119985723} & 0 \end{pmatrix},$$

and

$$T_2^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{4442006}{40595475} \\ 0 & 0 & 0 & 0 & 0 & \frac{45955496}{590161785} \\ 0 & 0 & 0 & 0 & 0 & \frac{3878126}{44681305} \\ 0 & 0 & 0 & 0 & 0 & -\frac{337472}{13331747} \\ 0 & 0 & 0 & 0 & 0 & -\frac{24813285}{853231808} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1560500}{119985723} \end{pmatrix}$$

$$Z_2^{(3)}Y_{C+2}=Z_1^{(3)}Y_{C+1}-v\sum_{i=1}^{2}\left(U_i^{(3)}F_{C+i}+vH_i^{(3)}G_{C+i}+vT_i^{(3)}Q_{C+i}\right) \quad (28)$$

The polynomials of D-and E-stability are constructed by introducing (26), (27) and (28) to (25) and (15), (18) and (21) to (25) as stated below

$$\lambda^{(1)}(\psi)=\det\left[\left(Z_2^{(1)}-P_1U_2^{(1)}-P_1H_2^{(1)}\right)\psi^{2+r}-\left(Z_1^{(1)}-P_1U_1^{(1)}-P_1H_1^{(1)}\right)\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(1)}-H_i^{(1)}-T_i^{(1)}\right)\psi^i\right] \quad (29)$$

$$\lambda^{(2)}(\psi)=\det\left[\left(Z_2^{(2)}-P_1U_2^{(2)}-P_1H_2^{(2)}\right)\psi^{2+r}-\left(Z_1^{(2)}-P_1U_1^{(2)}-P_1H_1^{(2)}\right)\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(2)}-H_i^{(2)}-T_i^{(2)}\right)\psi^i\right] \quad (30)$$

$$\lambda^{(3)}(\psi)=\det\left[\left(Z_2^{(3)}-P_1U_2^{(3)}-P_1H_2^{(3)}\right)\psi^{2+r}-\left(Z_1^{(3)}-P_1U_1^{(3)}-P_1H_1^{(3)}\right)\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(3)}-H_i^{(3)}-T_i^{(3)}\right)\psi^i\right] \quad (31)$$

and

$$\chi^{(1)}(\psi)=\det\left[Z_2^{(1)}\psi^{2+r}-Z_1^{(1)}\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(1)}-H_i^{(1)}-T_i^{(1)}\right)\psi^i\right] \quad (32)$$

$$\chi^{(2)}(\psi)=\det\left[Z_2^{(2)}\psi^{2+r}-Z_1^{(2)}\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(2)}-H_i^{(2)}-T_i^{(2)}\right)\psi^i\right] \quad (33)$$

$$\chi^{(3)}(\psi)=\det\left[Z_2^{(3)}\psi^{2+r}-Z_1^{(3)}\psi^{1+r}-P_2\sum_{i=1}^{2}\left(U_i^{(3)}-H_i^{(3)}-T_i^{(3)}\right)\psi^i\right] \quad (34)$$

Making use of Maple 18 and MATLAB, the region of D- and E-stability for (15), (18) and (21) are shown in Fig.1 to 6.

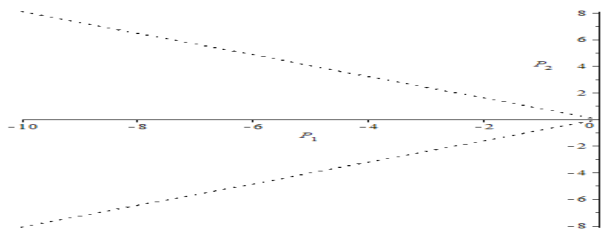

Fig.1.Region of D-stability (HESDBBDFM) in (15)

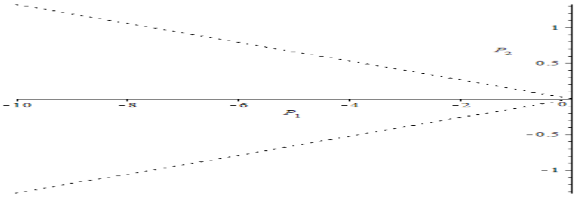

Fig.2.Region of D-stability (HESDBBDFM) in (18)

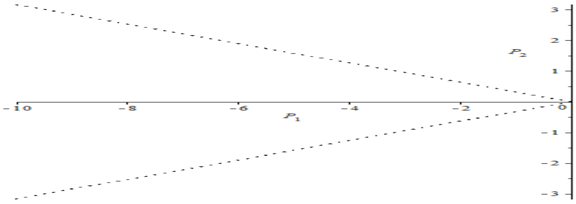

Fig.3.Region of D-stability (HESDBBDFM) in (21)

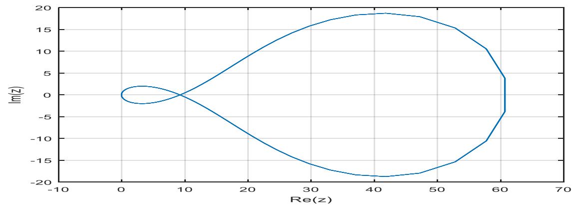

Fig.4. Region of E-stability (HESDBBDFM) in (15)

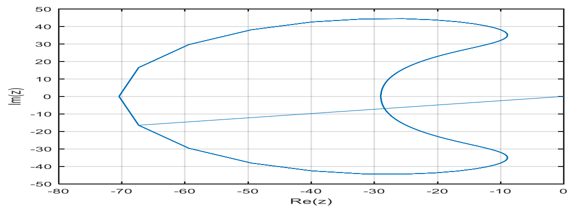

Fig.5. Region of E-stability (HESDBBDFM) in (18)

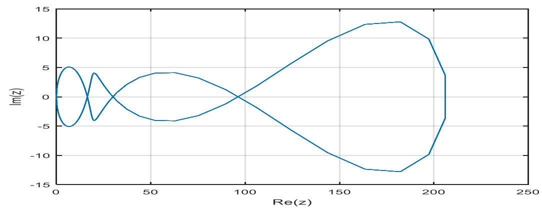

Fig.6. Region of E-stability (HESDBBDFM) in (21)

The D-stability regions in Figs 1 to 3 lie inside the open-ended region while the E-stability regions in Figs 4 to 6 lie inside the enclosed region.

V. NUMERICAL COMPUTATIONS

In this section, some first-order delay differential equations shall be solved using (15), (18) and (21) of the discrete schemes been constructed. The delay argument shall be evaluated using the idea of sequence developed by [14] and the time it took the Central Processing Unit (CPUT) to produce the numerical solutions of each step number of the proposed method shall be taken into consideration.

J. Implementation of Numerical Problems

Problem 1

$$x'(t)=-1000x(t)+997e^{-3}x(t-1)+(1000-997e^{-3}),\ 0\le t\le 3$$

$$x(t)=1+e^{-3t}, t\le 0$$

Exact solution $x(t)=1+e^{-3t}$

Problem 2

$$x'(t)=-1000x(t)+x(t-(\ln(1000-1))),\ 0\le t\le 3$$

$x(t) = e^{-t}, t \leq 0$

Exact solution $x(t) = e^{-t}$

These examples were solved using the discrete schemes in (15), (18) and (21), and the results are presented below

Table 4.1.1: Numerical Solution of HESDBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 1

t	Exact Solution	k = 2 Numerical Solution	k = 3 Numerical Solution	k = 4 Numerical Solution
0.1	1.740818221	1.970390437	1.970529152	1.970374348
0.2	1.548811636	1.941809031	1.941731461	1.941788851
0.3	1.40656966	1.91298104	1.913969594	1.913912749
0.4	1.301194212	1.886859792	1.886038111	1.886951839
0.5	1.22313016	1.860749741	1.860769502	1.859897788
0.6	1.165298888	1.834392143	1.835241935	1.835195767
0.7	1.122456428	1.810528715	1.81061804	1.810606525
0.8	1.090717953	1.786666041	1.785847578	1.786611636
0.9	1.067205513	1.762576913	1.763434101	1.763406681
1	1.049787068	1.740767469	1.740793138	1.74011964
1.1	1.036883167	1.718958629	1.718953706	1.718859636
1.2	1.027323722	1.696942804	1.69698428	1.697695504
1.3	1.020241911	1.677010486	1.677105302	1.677042908
1.4	1.014995577	1.657078712	1.657024573	1.657070219
1.5	1.011108997	1.636957757	1.637654735	1.63702689
1.6	1.008229747	1.618740997	1.618169611	1.618728224
1.7	1.006096747	1.600524726	1.60053853	1.600512087
1.8	1.004516581	1.582135549	1.582728521	1.58273623
1.9	1.003345965	1.565486689	1.565549016	1.565545579
2	1.002478752	1.548838275	1.548267259	1.548294101
2.1	1.001836305	1.532031842	1.532629896	1.532544316
2.2	1.001360368	1.516815925	1.516833835	1.516865542
2.3	1.001007785	1.501600416	1.501596981	1.501565723
2.4	1.000746586	1.486240497	1.486269397	1.48676959
2.5	1.000553084	1.472334187	1.472400342	1.471921127
2.6	1.000409735	1.458428262	1.45839049	1.458365141
2.7	1.000303539	1.444390376	1.444876613	1.444870294
2.8	1.000224867	1.431680945	1.431282303	1.431701618
2.9	1.000166586	1.418971884	1.418981516	1.418966469
3	1.00012341	1.406142215	1.406555894	1.406186286

CPU time of HESDBBDFM for k = 2 is 0.110s, k = 3 is 0.221s and k = 4 is 0.301s

Table 4.1.2: Numerical Solution of HESDBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 2

t	Exact Solution	k = 2 Numerical Solution	k = 3 Numerical Solution	k = 4 Numerical Solution
0.1	0.904837418	0.990040383	0.990064477	0.990037076
0.2	0.818730753	0.980206351	0.980192868	0.980203037
0.3	0.740818221	0.970280432	0.970452303	0.970442223
0.4	0.670320046	0.960778479	0.960632917	0.960795095
0.5	0.60653066	0.951237066	0.951241251	0.951082667
0.6	0.548811636	0.941602626	0.941759144	0.941750011
0.7	0.496585304	0.932383163	0.932400276	0.932398214
0.8	0.449328964	0.923123764	0.922966367	0.923113132
0.9	0.40656966	0.913774054	0.913942554	0.913936594
1	0.367879441	0.904827075	0.90483224	0.904697588
1.1	0.332871084	0.895841334	0.895840338	0.895820317
1.2	0.301194212	0.886767954	0.88677632	0.886924617
1.3	0.272531793	0.878085393	0.878106353	0.878092374
1.4	0.246596964	0.869365222	0.869353259	0.86936338
1.5	0.22313016	0.860559992	0.860713936	0.860574961
1.6	0.201896518	0.852134055	0.852005342	0.852130645
1.7	0.182683524	0.843671595	0.843675311	0.843668792
1.8	0.165298888	0.835126616	0.835265431	0.835267303
1.9	0.149568619	0.826949684	0.82696486	0.826964027
2	0.135335283	0.818737332	0.818597718	0.818604245
2.1	0.122456428	0.810444887	0.810594329	0.810571743
2.2	0.110803158	0.802509628	0.802514204	0.802522581
2.3	0.100258844	0.794539987	0.794539104	0.794530836
2.4	0.090717953	0.786492617	0.786500058	0.786632516
2.5	0.082084999	0.778791882	0.77881047	0.77868044
2.6	0.074273578	0.771057782	0.771047173	0.771039693
2.7	0.067205513	0.763248259	0.76338478	0.763383093
2.8	0.060810063	0.755775107	0.755660941	0.75578111
2.9	0.05502322	0.74826958	0.748272876	0.748267995
3	0.049787068	0.740690857	0.740813981	0.740703736

CPU time of HESDBBDFM for k = 2 is 0.112s, k = 3 is 0.225s and k = 4 is 0.305s

VI. RESULTS AND DISCUSSIONS

Here, the numerical solutions obtained after solving some first order DDE using the schemes derived in (15), (18) and (21) shall be analyzed by computing their absolute errors.

K. Analysis of Results

The analysis of results is obtained by determining absolute differences of the exact solutions and the numerical solutions. The results are presented in the tables 5.1.1 to 5.1.2 while the comparison between the proposed method and other existing methods are presented in tables 5.1.3 to 5.1.4 and Figs. 7 to 8.

Table 5.1.1: Absolute Error of HESDBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 1

t	k = 2 Error	k = 3 Error	k = 4 Error
0.1	0.229572216	0.229710931	0.229556127
0.2	0.392997395	0.392919825	0.392977215
0.3	0.50641138	0.507399934	0.507343089
0.4	0.58566558	0.584843899	0.585757627
0.5	0.637619581	0.637639342	0.636767628
0.6	0.669093255	0.669943047	0.669896879
0.7	0.688072287	0.688161612	0.688150097
0.8	0.695948088	0.695129625	0.695893683
0.9	0.6953714	0.696228588	0.696201168
1	0.690980401	0.69100607	0.690332572
1.1	0.682075462	0.682070539	0.681976469
1.2	0.669619082	0.669660558	0.670371782
1.3	0.656768575	0.656863391	0.656800997
1.4	0.642083135	0.642028996	0.642074642
1.5	0.62584876	0.626545738	0.625917893
1.6	0.61051125	0.609939864	0.610498477
1.7	0.594427979	0.594441783	0.59441534
1.8	0.577618968	0.57821194	0.578219649
1.9	0.562140724	0.562203051	0.562199614
2	0.546359523	0.545788507	0.545815349
2.1	0.530195537	0.530793591	0.530708011
2.2	0.515455557	0.515473467	0.515505174
2.3	0.500592631	0.500589196	0.500557938
2.4	0.485493911	0.485522811	0.486023004

2.5	0.471781103	0.471847258	0.471368043
2.6	0.458018527	0.457980755	0.457955406
2.7	0.444086837	0.444573074	0.444566755
2.8	0.431456078	0.431057436	0.431476751
2.9	0.418805298	0.41881493	0.418799883
3	0.406018805	0.406432484	0.406062876

Table 5.1.2: Absolute Error of HESDBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 2

t	k = 2 Error	k = 3 Error	k = 4 Error
0.1	0.085202965	0.085227058	0.085199658
0.2	0.161475598	0.161462115	0.161472284
0.3	0.229462211	0.229634083	0.229624002
0.4	0.290458433	0.290312871	0.290475049
0.5	0.344706406	0.344710591	0.344552007
0.6	0.39279099	0.392947508	0.392938375
0.7	0.435797859	0.435814972	0.435812911
0.8	0.4737948	0.473637403	0.473784168
0.9	0.507204394	0.507372894	0.507366934
1	0.536947634	0.536952798	0.536818147
1.1	0.56297025	0.562969255	0.562949233
1.2	0.585573742	0.585582108	0.585730405
1.3	0.6055536	0.60557456	0.605560581
1.4	0.622768258	0.622756295	0.622766416
1.5	0.637429832	0.637583776	0.637444801
1.6	0.650237537	0.650108824	0.650234127
1.7	0.660988071	0.660991786	0.660985268
1.8	0.669827728	0.669966542	0.669968414
1.9	0.677381064	0.677396241	0.677395408
2	0.683402048	0.683262435	0.683268962
2.1	0.687988459	0.688137901	0.688115315
2.2	0.69170647	0.691711046	0.691719422
2.3	0.694281143	0.69428026	0.694271992
2.4	0.695774664	0.695782105	0.695914563
2.5	0.696706883	0.696725472	0.696595441
2.6	0.696784203	0.696773595	0.696766115
2.7	0.696042746	0.696179267	0.69617758
2.8	0.694965044	0.694850879	0.694971047
2.9	0.69324636	0.693249656	0.693244775
3	0.690903789	0.691026912	0.690916667

The notations used in the table below are stated as
HESDBBDFM = Hybrid Extended Second Derivative Block Backward Differentiation Formulae Methods for step numbers $k = 2, 3$ and 4.

ESDBBDFM = Second Derivative Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ and 4.

SDBBDFM = Extended Second Derivative Block Backward Differentiation Formulae Methods for step numbers $k = 2, 3$ and 4.

RBBDFM = Reformulated Block Backward Differentiation Formulae Methods for step numbers $k = 3$ and 4.

MAXE = Maximum Error. The maximum error (MAXE) is the highest value of the absolute error for total number of steps taken and the numerical method with the highest Maximum Error is the method that performs better.

Table 5.1.3: Comparison between the Maximum Absolute Errors of HESDBBBDFM $k = 2, 3$ and 4 and [7, 10, 14] of the same Example 1 for constant step size v = 0.01.

Numerical Method	COMPARED MAXEs with [7,10,14]
HESDBBBDFM MAXE for k = 2	0.695948088
HESDBBBDFM MAXE for k = 3	0.695129625
HESDBBBDFM MAXE for k = 4	0.695893683
ESDBBBDFM MAXE for k = 2	5.04E-02
ESDBBBDFM MAXE for k = 3	6.69E-02
ESDBBBDFM MAXE for k = 4	7.09E-02
SDBBDFM MAXE for k = 2	5.42E-06
SDBBDFM MAXE for k = 3	0.000243476
SDBBDFM MAXE for k = 4	0.001368661
RBBDFM MAXE for k = 3	1.54E-09
RBBDFM MAXE for k = 4	1.04E-09

Table 5.1.4: Comparison between the Maximum Absolute Errors of HESDBBBDFM $k = 2, 3$ and 4 and [7, 10, 14] of the same Example 2 for constant step size v = 0.01.

Numerical Method	COMPARED MAXEs with [7,10,14]
HESDBBBDFM MAXE for k = 2	0.696784203
HESDBBBDFM MAXE for k = 3	0.696773595
HESDBBBDFM MAXE for k = 4	0.696766115
ESDBBBDFM MAXE for k = 2	2.08E-03
ESDBBBDFM MAXE for k = 3	1.91E-03
ESDBBBDFM MAXE for k = 4	3.43E-03
SDBBDFM MAXE for k = 2	5.42E-06
SDBBDFM MAXE for k = 3	0.000243476
SDBBDFM MAXE for k = 4	0.001368661
RBBDFM MAXE for k = 3	4.88E-06
RBBDFM MAXE for k = 4	4.38E-06

Making use of Microsoft Excel, the compared MAXE for HESDBBBDFM with [7, 10, 14] for Example 1 and 2 are shown in Fig.7 and 8 as follows,

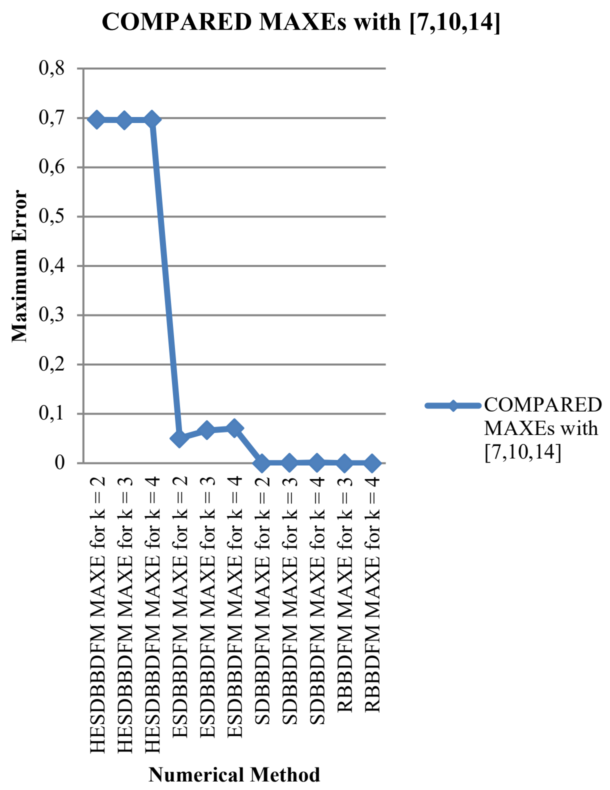

Fig.7. Compared MAXE for HESDBBBDFM with [7, 10, 14] for Example 1

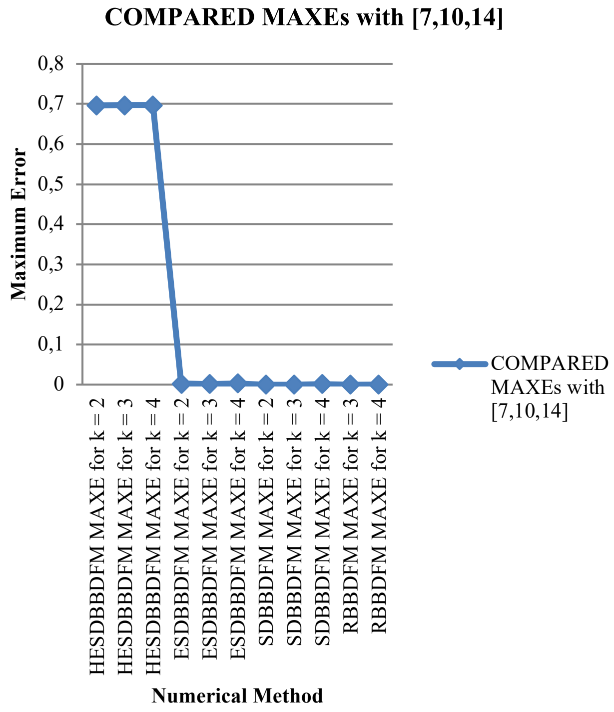

Fig.8. Compared MAXEs for HESDBBDFM with [7, 10, 14] for Example 2

VII. CONCLUSIONS

In conclusion, the discrete schemes of (15), (18) and (21), were worked-out from their individual continuous form and were revealed to be convergent, D - and E-stable. Also, it was revealed in tables 5.1.1 to 5.1.2 that the lower step number of HESDBBDF method with integrated off-grid extended future points and one extended future point performed better and faster than the higher step numbers of HESDBBDF method with integrated off-grid extended future points and one extended future point when compared with the exact solutions. Comparing HESDBBDFM with other existing methods, it proved its superiority in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at constant step width v as presented in table 5.1.3 to 5.1.4 and figure 7 to 8. Therefore, it is recommended that HESDBBDFM schemes for step numbers $k = 2, 3$,and 4 are suitable for solving DDEs. Further studies should be investigated for step numbers $k = 5, 6, 7...$ on the formulation of discrete schemes of HESDBBDFM for numerical solutions of DDEs without the application of interpolation techniques in investigating the delay arguments.

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