Conformal Fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$ Expansion Method and Its Applications for Solving the Nonlinear Fractional Sharma-Tasso-Olver equation

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***Abstract* — In this article, we generalize the $\left(\frac{G'(\xi)}{G(\xi)}\right)-$ expansion method which is one of the most important methods to finding the exact solutions of nonlinear partial differential equations. The new generalized method, named conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$expansion method, takes advantage of Katugampola's fractional derivative to create many useful traveling wave solutions of the nonlinear conformal fractional Sharma-Tasso-Olver equation. The obtained solutions have been articulated by the hyperbolic, trigonometric and rational functions with arbitrary constants. These solutions are algebraically verified using Maple and their physical characteristics are illustrated in some special cases.**

***Index Terms* — Conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$expansion method, nonlinear conformal fractional Sharma-Tasso-Olver equation.**

I. Introduction

Since ancient times, mathematics and physics have been two sides of the same coin. They work together to find effective solutions to many real-world problems and to explain many physical phenomena.

In the past, a wide range of methods have been developed to generate analytical solutions of nonlinear partial differential equations. Among these methods are the $\left(\frac{G'(\xi)}{G(\xi)}\right)-$ expansion method [1], [2], the double auxiliary equations method [3], the generalized of $\exp\left(-\phi(\xi)\right)-$expansion method [4], [5], and various other methods [6]-[11].

Several years ago, many mathematicians proposed formulas of fractional derivatives that physicists use for modeling many phenomena. For example, in [12]-[15] Jumarie proposed a modified Riemann-Liouville fractional derivative as follows:

$$f^{(\alpha)}(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-x)^{-\alpha}\left(f(x)-f(0)\right)dx. \tag{1}$$

and the chain rule defined in (2):

$$\left(f\left(u(t)\right)\right)^{(\alpha)}=f_u{}'u^{(\alpha)}(t), \tag{2}$$

The chain rule has been applied by several authors to find the exact solutions to some nonlinear fractional differential equations. For example, Zhang and Zhang [16] proposed the fractional sub-equation method to search for exact solutions of nonlinear time fractional biological population. Jafari *et al.* [17] used this method to obtain the exact solutions of the nonlinear fractional Sharma-Tasso-Olver equation, while Cesar *et al.* [18] solved it using the improved generalized tanh-coth method.

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This paper proposes a new conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$ expansion method, which generalizes the $\left(\frac{G'(\xi)}{G(\xi)}\right)-$ expansion method, using the definition of the new conformal fractional derivation presented by Katugampola [21]. The new method satisfies the product, quotient, and chain rules for obtaining the exact traveling wave solutions for the nonlinear conformal fractional Sharma-Tasso-Olver equation [17], [18]:

$$D_t^\alpha u + 3\delta u^2 D_x^\alpha u + 3\delta (D_x^\alpha u)^2 + 3\delta u D_x^{2\alpha} u + \delta D_x^{3\alpha} u = 0, \tag{3}$$

where $0 < \alpha \leq 1, u = u(x,t), t > 0, \delta$ is a constant.

The new obtained solutions of the nonlinear conformal fractional Sharma-Tasso-Olver equation are algebraically verified in Section 3 using Maple and illustrated with their physical characteristics in Section 4 in some special cases.

II. THE GENERAL EXPRESSION FOR CONFORMAL FRACTIONAL $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$ −EXPANSION METHOD

To understand the conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$−expansion method, we first provide the main definition and properties of the conformal fractional calculus proposed by Katugampola in [21].

Definition 2.1. Let $f: [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function. The conformal fractional derivative of f of order α is defined by,

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \left(f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)\right)/\varepsilon \tag{4}$$

for all $t > 0, \alpha \in (0,1]$ .If f is α −differentiable in some $(0,a), a > 0$, and $\lim_{t \to 0^+} D^\alpha(f)(t)$ exist, then define:

$$D^\alpha(f)(t) = \lim_{t \to 0^+} D^\alpha(f)(t) \tag{5}$$

Theorem 2.1. Let $\alpha \in (0,1]$ and f, g be two α −differentiable functions at a point $t > 0$ .Then

- **i.** $D_t^\alpha(af + bg)(t) = aD_t^\alpha f(t) + bD_t^\alpha g(t)$, for all $a, b \in \mathbb{R}$.
- **ii.** $D_t^\alpha(t^n) = nt^{n-\alpha}$, for all $n \in \mathbb{R}$.
- **iii.** $D_t^\alpha(c) = 0$, for all constant functions $f(t) = c$.
- **iv.** $D_t^\alpha(fg)(t) = D_t^\alpha f(t)g(t) + D_t^\alpha g(t)f(t)$.
- **v.** $D_t^\alpha\left(\frac{f}{g}\right)(t) = \left(D_t^\alpha f(t)g(t) - D_t^\alpha(g)f(t)\right)/g(t)^2$
- **vi.** $D_t^\alpha(f \circ g)(t) = D_g^\alpha f\left(g(t)\right)D_t^\alpha g(t)$.

The main steps of the new proposed conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$ expansion method is described as follows:

Step 1. Consider a conformal fractional partial differential equation in the form:

$$F(u, u_t, u_x, D_t^\alpha u, D_x^\alpha u, \dots) = 0, \qquad 0 < \alpha \leq 1, \tag{6}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Katugampola's fractional derivative of u. Using the transformation,

$$u(x,t) = u(\xi) \ , \quad \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha, \tag{7}$$

Equation (6) reduced to a nonlinear conformal fractional ordinary differential equation for $u = u(\xi)$ in the form:

$$P\left(u, D_\xi^\alpha u, D_\xi^{2\alpha} u, D_\xi^{3\alpha} u, \dots\right) = 0. \tag{8}$$

Step 2. By balancing the highest derivative and nonlinear terms in (8), and using the (9), the value of the positive integer (m) is determined:

$$Degree\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = mp + s(m+q), \tag{9}$$

Step 3. The solution of (8) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^m \alpha_i \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^i, \tag{10}$$

where, α_i $(i = 0,1,\dots,m)$ are constants to be determined, and $G(\xi)$ satisfies the following conformal fractional differential equation:

$$D_\xi^{2\alpha} G(\xi) + \lambda D_\xi^\alpha G(\xi) + \mu G(\xi) = 0 \tag{11}$$

First, we can express the solutions of the conformal fractional equation (11) as follows:

$$G(\xi) = \begin{cases} \left(C_1 \sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right)\right) e^{\left(\frac{-\lambda\xi^\alpha}{2\alpha}\right)}; (\lambda^2-4\mu) > 0 \\ \left(C_1 \sin\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right) + C_2 \cos\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right)\right) e^{\left(\frac{-\lambda\xi^\alpha}{2\alpha}\right)}; (\lambda^2-4\mu) < 0 \\ (C_1 + C_2\xi^\alpha) e^{\left(\frac{-\lambda\xi^\alpha}{2\alpha}\right)}\ ; (\lambda^2-4\mu) = 0 \end{cases} \tag{12}$$

Thus, the $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$can be reformulated as follows:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} -\frac{\lambda}{2} + \frac{\alpha}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\left(\frac{C_1\cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right) + C_2\sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right)}{C_1\sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right) + C_2\cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^\alpha\right)}\right); (\lambda^2-4\mu) > 0 \\ -\frac{\lambda}{2} + \frac{\alpha}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\left(\frac{C_1\cos\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right) - C_2\sin\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right)}{C_1\sin\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right) + C_2\cos\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^\alpha\right)}\right); (\lambda^2-4\mu) < 0 \\ -\frac{\lambda}{2} + \frac{C_2\alpha}{C_1 + C_2\xi^\alpha}; (\lambda^2-4\mu) = 0 \end{cases} \tag{13}$$

where $C_1, C_2$are arbitrary constants.

Step 4. Substituting (10) into (8) and using (11), and then setting all the coefficients of $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^i$ in the resulting systems to zero, yields a system of algebraic equations for k_1, k_2, λ, μ and α_i $(i = 0,\dots,m)$. By solving this system and substituting $k_1, k_2, \lambda, \mu, \alpha_i$ and the formula (13) into (10), we obtain the exact solution for (8).

III. The Exact Solution For nonlinear conformal fractional Sharma-Tasso-Olver equation

In this section, the conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$ expansion method will be applied to find the exact solutions of the nonlinear conformal fractional Sharma-Tasso-Olver equation:

$$D_t^\alpha u + 3\delta u^2 D_x^\alpha u + 3\delta (D_x^\alpha u)^2 + 3\delta u D_x^{2\alpha} u + \delta D_x^{3\alpha} u = 0, \tag{14}$$

where $0 < \alpha \leq 1$.

Suppose that,

$$u(x,t) = u(\xi) \ , \quad \xi = \left(\frac{k_1}{\alpha}\right) x^{\alpha} + \left(\frac{k_2}{\alpha}\right) t^{\alpha}, \tag{15}$$

where k_1, k_2 are a constants. Substituting (15) into (14), gives the following nonlinear conformal fractional ordinary differential equations:

$$k_2 D_{\xi}^{\alpha} u + 3\delta k_1 u^2 D_{\xi}^{\alpha} u + 3\delta k_1^2 \left(D_{\xi}^{\alpha} u\right)^2 + 3\delta k_1^2 u D_{\xi}^{2\alpha} u + \delta k_1^3 D_{\xi}^{3\alpha} u = 0. \tag{16}$$

Suppose that (16) has the following solution:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{D_{\xi}^{\alpha} G(\xi)}{G(\xi)}\right)^i, \alpha_i \in \mathbb{R} \tag{17}$$

Balancing the order of $D_{\xi}^{3\alpha} u$ and $u^2 D_{\xi}^{\alpha} u$, we find $m = 1$. So,

$$u(\xi) = \alpha_0 + \alpha_1 \left(\frac{D_{\xi}^{\alpha} G(\xi)}{G(\xi)}\right). \tag{18}$$

Substituting (18) and (11) into (16), the left-hand side is converted into polynomials in $\left(\frac{D_{\xi}^{\alpha} G(\xi)}{G(\xi)}\right)^j$, $(j = 0,1,2,\dots.)$. By collecting each coefficient of these resulted polynomials to zero, we obtain a system of algebraic equations for α_0, α_1, k_1 and k_2, which are not presented for sake of clarity. By Solving these algebraic equations with the help of algebraic software Maple, we obtain:

Case 1

$$\left\{\alpha_0 = \frac{3\delta\lambda k_1^2 + \sqrt{-3\delta k_1\left(4k_2 + k_1^3 \delta(\lambda^2 - 4\mu)\right)}}{6\delta k_1}, \alpha_1 = k_1, k_1 = k_1, k_2 = k_2\right\} \tag{19}$$

Substituting (19) into (18), we have:

$$\begin{cases} u(\xi) = \left(\dfrac{3\delta\lambda k_1^2 + \sqrt{-3\delta k_1\left(4k_2 + k_1^3 \delta(\lambda^2 - 4\mu)\right)}}{6\delta k_1}\right) + k_1 \left(\dfrac{D_{\xi}^{\alpha} G(\xi)}{G(\xi)}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^{\alpha} + \left(\frac{k_2}{\alpha}\right) t^{\alpha}. \end{cases} \tag{20}$$

Consequently, the exact solution of the of the nonlinear conformal fractional Sharma-Tasso-Olver equation (14) with the help of (13), are obtained in the followin form:

Case (1-1). When $(\lambda^2 - 4\mu) > 0$,

$$\begin{cases} u_{1,1}(\xi) = \left(\begin{array}{l} \dfrac{3\delta\lambda k_1^2 + \sqrt{-3\delta k_1\left(4k_2 + k_1^3 \delta(\lambda^2 - 4\mu)\right)}}{6\delta k_1} - \dfrac{\lambda k_1}{2} \\ + \dfrac{\alpha k_1}{2} \sqrt{\dfrac{(\lambda^2 - 4\mu)}{\alpha^2}} \left(\dfrac{C_1 \cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^{\alpha}\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^{\alpha}\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^{\alpha}\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2-4\mu)}{\alpha^2}}\xi^{\alpha}\right)}\right) \end{array}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^{\alpha} + \left(\frac{k_2}{\alpha}\right) t^{\alpha}. \end{cases} \tag{21}$$

Case (1-2). When $(\lambda^2 - 4\mu) < 0$,

$$\begin{cases} u_{1,2}(\xi) = \left(\begin{array}{l} \left(\dfrac{3\delta\lambda k_1^2 + \sqrt{-3\delta k_1\left(4k_2 + k_1^3 \delta(\lambda^2 - 4\mu)\right)}}{6\delta k_1}\right) - \dfrac{\lambda k_1}{2} \\ + \dfrac{\alpha k_1}{2} \sqrt{\dfrac{(4\mu - \lambda^2)}{\alpha^2}} \left(\dfrac{C_1 \cos\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^{\alpha}\right) - C_2 \sin\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^{\alpha}\right)}{C_1 \sin\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^{\alpha}\right) + C_2 \cos\left(\frac{1}{2}\sqrt{\frac{(4\mu-\lambda^2)}{\alpha^2}}\xi^{\alpha}\right)}\right) \end{array}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^{\alpha} + \left(\frac{k_2}{\alpha}\right) t^{\alpha}. \end{cases} \tag{22}$$

Case (1-3). When $(\lambda^2 - 4\mu) = 0$,

$$\begin{cases} u_{1,3}(\xi) = \left(\frac{3\delta\lambda k_1^2 + \sqrt{-12\delta k_1 k_2}}{6\delta k_1}\right) - \frac{\lambda k_1}{2} + \frac{C_2 \alpha k_1}{C_1 + C_2 \xi^\alpha}, \\ \xi = \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha. \end{cases} \tag{23}$$

Case (2).

$$\{\alpha_0 = \lambda k_1, \alpha_1 = 2k_1, k_1 = k_1, k_2 = -\delta k_1^3(\lambda^2 - 4\mu)\} \tag{24}$$

Substituting (24) into (18), we have:

$$\begin{cases} u(\xi) = \lambda k_1 + 2k_1 \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha - \left(\frac{\delta k_1^3(\lambda^2 - 4\mu)}{\alpha}\right) t^\alpha. \end{cases} \tag{25}$$

Consequently, the exact solution of the of the nonlinear conformal fractional Sharma-Tasso-Olver equation (14) with the help of (13), are obtained in the followin form:

Case (2-1). When $(\lambda^2 - 4\mu) > 0$,

$$\begin{cases} u_{2,1}(\xi) = \alpha k_1 \sqrt{\frac{(\lambda^2 - 4\mu)}{\alpha^2}} \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2 - 4\mu)}{\alpha^2}} \xi^\alpha\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2 - 4\mu)}{\alpha^2}} \xi^\alpha\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2 - 4\mu)}{\alpha^2}} \xi^\alpha\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\frac{(\lambda^2 - 4\mu)}{\alpha^2}} \xi^\alpha\right)}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha - \left(\frac{\delta k_1^3(\lambda^2 - 4\mu)}{\alpha}\right) t^\alpha. \end{cases} \tag{26}$$

Case (2-2). When $(\lambda^2 - 4\mu) < 0$,

$$\begin{cases} u_{2,2}(\xi) = \alpha k_1 \sqrt{\frac{(4\mu - \lambda^2)}{\alpha^2}} \left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{\frac{(4\mu - \lambda^2)}{\alpha^2}} \xi^\alpha\right) - C_2 \sin\left(\frac{1}{2}\sqrt{\frac{(4\mu - \lambda^2)}{\alpha^2}} \xi^\alpha\right)}{C_1 \sin\left(\frac{1}{2}\sqrt{\frac{(4\mu - \lambda^2)}{\alpha^2}} \xi^\alpha\right) + C_2 \cos\left(\frac{1}{2}\sqrt{\frac{(4\mu - \lambda^2)}{\alpha^2}} \xi^\alpha\right)}\right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha - \left(\frac{\delta k_1^3(\lambda^2 - 4\mu)}{\alpha}\right) t^\alpha. \end{cases} \tag{27}$$

Case (2-3). When $(\lambda^2 - 4\mu) = 0$,

$$\begin{cases} u_{2,3}(\xi) = \frac{2\alpha k_1 C_2}{C_1 + C_2 \xi^\alpha}, \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha. \end{cases} \tag{28}$$

IV. Physical Explanation and Interpretations of the Solutions

In this section, physical representation of the obtained exact and solitary wave solution to nonlinear conformal fractional Sharma-Tasso-Olver equation (14). shall be discussed these solutions are graphically represented and their kind of solution verified. solutions $u_{1,1}(\xi), u_{1,3}(\xi)$ and $u_{2,1}(\xi)$ are of the kink type soliton solution (Fig. 1 only shows the shape of $u_{1,1}(\xi)$ with $k_1 = -\sqrt{2}, k_2 = 4, \delta = \sqrt{2}, C_1 = \mu = 1, C_2 = 2, \lambda = 2\sqrt{2}, \alpha = \frac{1}{2}$, Fig. 3 only shows the shape of $u_{1,3}(\xi)$ with $k_1 = C_1 = \mu = 1, k_2 = -1, \delta = 2\sqrt{3}, C_2 = \lambda = 2, \alpha = \frac{1}{2}$ and Fig.4 only shows the shape of $u_{2,1}(\xi)$ with $k_1 = \frac{1}{2}, \delta = C_1 = \mu = 1, C_2 = 2, \lambda = 2\sqrt{2}, \alpha = \frac{1}{2}$).

Solutions $u_{1,2}(\xi), u_{2,2}(\xi)$, are the multiple bright and dark solitons solution (Fig. 2 only shows the shape of $u_{1,2}(\xi)$ with $k_1 = -1, k_2 = C_1 = 1, \delta = \sqrt{3}, C_2 = \lambda = \mu = 2, \alpha = \frac{1}{2}$, and Fig. 5 only shows the shape of $u_{2,2}(\xi)$ with $k_1 = C_1 = 1, \delta = \frac{1}{2}, C_2 = e, \lambda = \mu = 2, \alpha = \frac{1}{2}$).

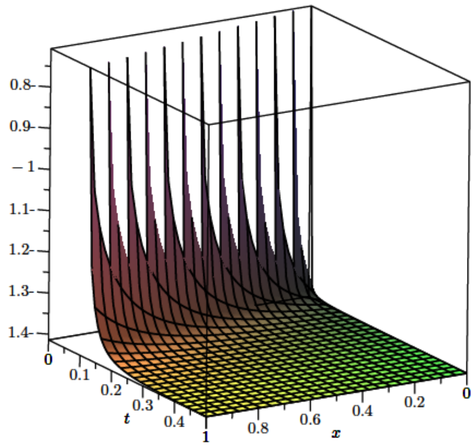

Fig. 1.
$u_{1,1}(\xi)$ with $k_1 = -\sqrt{2}, k_2 = 4, \delta = \sqrt{2}, C_1 = \mu = 1, C_2 = 2, \lambda = 2\sqrt{2}, \alpha = \frac{1}{2}$.

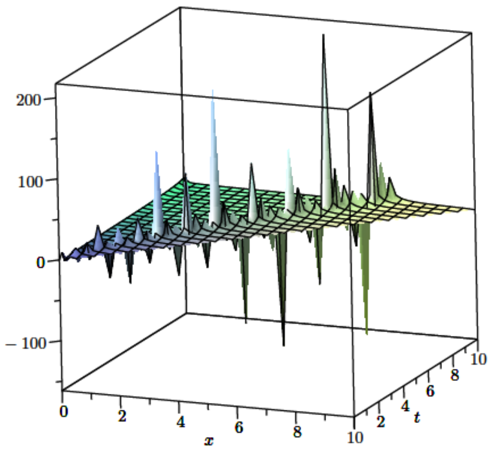

Fig. 2.
$u_{1,2}(\xi)$ with $k_1 = -1, k_2 = C_1 = 1, \delta = \sqrt{3}, C_2 = \lambda = \mu = 2, \alpha = \dfrac{1}{2}$.

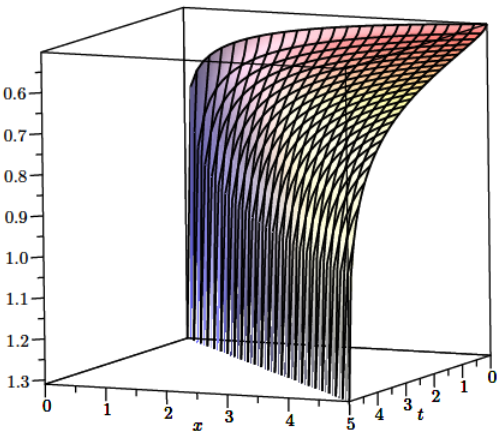

Fig. 3.
$u_{1,3}(\xi)$ with $k_1 = C_1 = \mu = 1, k_2 = -1, \delta = 2\sqrt{3}, C_2 = \lambda = 2, \alpha = \frac{1}{2}$.

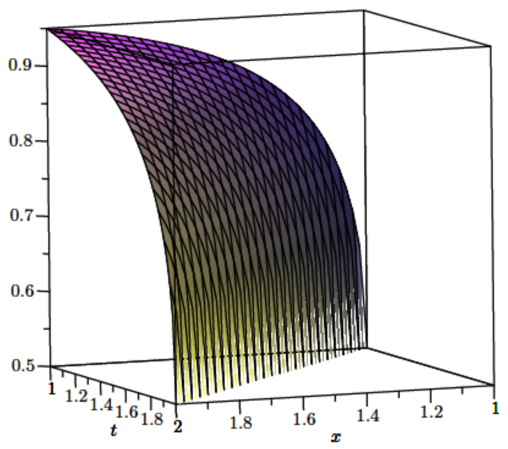

Fig. 4.
$u_{2,1}(\xi)$ with $k_1 = \frac{1}{2}, \delta = C_1 = \mu = 1, C_2 = 2, \lambda = 2\sqrt{2}, \alpha = \frac{1}{2}$.

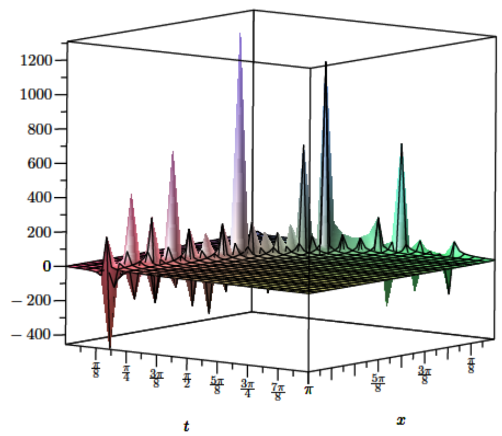

Fig. 5.
$u_{2,2}(\xi)$ with $k_1 = C_1 = 1, \delta = \frac{1}{2}, C_2 = e, \lambda = \mu = 2, \alpha = \frac{1}{2}$.

REMARK, all solutions of this article have been checked with maple by putting them back into the original equation (16).

V. CONCLUSION

In this paper, several useful exact solutions of the nonlinear conformal fractional Sharma-Tasso-Olver equation are obtained by using the new conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$ expansion method. These solutions are algebraically verified using Maple and illustrated with their physical characteristics in some special cases, to show their ability to explain some physical phenomena. The new conformal fractional $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)-$ expansion method is direct, effective, and can be used to solve many other NFPDEs in mathematical physics.

VI. DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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