New Series for Construction of Second Order Rotatable Designs

N. Ch. Bhatra Charyulu, A. Saheb Shaik, and G. Jayasree

***Abstract* — Second order Rotatable designs have good significance in response surface methodology. In this paper, two new seriesfor the construction the same using Binary Ternary Designs were presented with illustrated examples.**

***Keywords* — Balanced incomplete block design, binary ternary design, second order rotatable design.**

I. Introduction

The variance of estimated response at the uth design point of a second order response surface design model when $\Sigma x^2_{ui}=N\lambda_2$; $\Sigma x^4_{ui}=CN\lambda_4$; $\Sigma x^2_{ui}x^2_{uj}=N\lambda_4$ (the summation is over the design points u = 1, 2, …N) and all other moments of odd power summations in moment matrix are vanishes, is

$$V(\hat{Y}_u) = V(\hat{\beta}_0)+[V(\hat{\beta}_i) + 2\,\mathrm{Cov}\,(\hat{\beta}_0,\hat{\beta}_{ii})]\,\rho^2 + V(\hat{\beta}_{ii})\rho^4 + \left[\frac{(c-3)}{(c-1)N\lambda_4}\right]\Sigma x^2_{ui}\, x^2_{uj}$$

where $\sum_{i=1}^{v} x_{ui}^2 = \rho^2$ distance between the point to origin. When c = 3 it can be expressed as a function of ρ^2 as

$$V(\hat{Y}_u) = \alpha\,\rho^4 + \beta\,\rho^2 + \gamma.$$

Two new series for the construction of Second Order Rotatable Designs using Balanced Ternary Designs were constructed and presented in the following section.

II. Methodology

A series of Second Order Rotatable Designs for v factors can be constructed using the Balanced Ternary Designs of Nigam (1974). The detailed procedure is presented below.

Step 1: Let N_1 be the incidence matrix of a Balanced Incomplete Block Design with parameters v, b, r, k, λ. Form the blocks by adding the elements of all possible pairs of

i) (i, i′) (i ≤ i′) of rows of N_1, produces a balanced ternary design with parameters V = v, B = b + ${}^bC_2$, R = r(b+1), K = 2k, and π= (b+2)λ + r^2.

ii) (i, i′) (i<i′) of rows of N_1 (assume v ≥ 2k), produces a Balanced Ternary Design with parameters V =v, B =${}^bC_2$, R =r(b-1), K=2k, and π =(b-2)λ+ r^2.

Step 2: Replace the elements in the balanced ternary design, 2 with α, 1 with β, and associate each block with an appropriate fraction of factorials (say 2^{k_1}) with levels ±1 so that, no lower order interaction effects are confounded.

Step 3: Add n_0 central points (0, 0, … 0), where n_0 > 0 to the resulting design. The numbers of design points are

i) $n = 2^{k_1}(b + {}^bC_2) + n_0$; ii) $n = 2^{k_1}\,.\,{}^bC_2 + n_0$.

Step 4: Let t = α^2 / β^2. Select the real values for ‘t’ as

i) $$t = \frac{6\lambda(r-1) \pm \sqrt{36\lambda^2(r-1)^2 - 4({}^{r+1}c_2 - 3\lambda)\{r(b-1) - 3[(r-\lambda)^2 + \lambda(b-2r) + \lambda]\}}}{2({}^{r+1}c_2 - 3\lambda)}$$

Published on March 8, 2022.
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ii) $t = \frac{6\lambda(r-\lambda)\sqrt{36\lambda^2(r-\lambda)^2 - 4({}^{r}c_2 - 3^{\lambda}c_2)\{r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]\}}}{2({}^{r}c_2 - 3^{\lambda}c_2)}$

Choose the value for 'β', then evaluate the value for 'α' using $\alpha^2 = t\beta^2$. The resulting design provides a v-dimensional Second Order Rotatable Design with five levels (± α, ± β, 0).

III. Findings

THEOREM-1: A Second Order Rotatable Design with five levels (±α, ±β, 0) can be constructed using Balanced Ternary Design with parameters V = v, B = b + $^{b}C_2$, R = r(b+1), K = 2k, and π = (b+2)λ + r^2.

Proof: Let N_{BXV} be the incidence matrix of a Balanced Ternary Design with parameters V = v, B = b + $^{b}C_2$, R = r(b+1), K = 2k, and π = (b+2)λ + r. Each column of the incidence matrix has the elements 2's, 1's and 0's are repeated $^{r+1}C_2$, r(b-r) and $^{(b-r+1)}C_2$ times respectively. Every pair of columns contains the pairs (2,2), (1,1) and ((2,1) or (1,2)) occurs λ, $[(r-\lambda)^2 + \lambda(b-2r+\lambda)]$ and 2λ(r-λ) times respectively. Replace 2 with α and 1 with β. Associate each block with the levels ±1 by taking an appropriate fraction of factorials (say 2^{k_1}) for v factors,. After augmenting n_0 central points, the resulting design has n = 2^{k_1}(b + $^{b}C_2$) + n_0 design points. From the rotatability condition can obtained.

$$2^{k_1}\{ {}^{(r+1)}C_2\,\alpha^4 + r(b-1)\,\beta^4\} = 3.\,2^{k_1}\{\lambda\alpha^4 + [(r-\lambda)^2 + \lambda(b2r+\lambda)]\beta^4 + 2\lambda(r-\lambda)\,\alpha^2\beta^2\} \quad (1)$$

From the rotatable condition, we obtain

$$({}^{(r+1)}C_2 - 3\lambda)\alpha^4 + \{r(b-1) - 3[(r-\lambda)^2 + \lambda(b-2r) + \lambda]\}\beta^4 - (6\lambda(r-\lambda))\alpha^2\beta^2 = 0 \quad (2)$$

Let $t = \alpha^2/\beta^2$, then (2) can be expressed in the quadratic form as

$$({}^{(r+1)}C_2 - 3\lambda)\,t^2 - 6\lambda(r-1)t + \{r(b-1) - 3[(r-\lambda)^2 + \lambda(b-2r) + \lambda]\} = 0 \quad (2a)$$

Solving the quadratic equation in 't', we have,

$$t = \frac{6\lambda(r-1) \pm \sqrt{36\lambda^2(r-1)^2 - 4({}^{r+1}c_2 - 3\lambda)\{r(b-1) - 3[(r-\lambda)^2 + \lambda(b-2r) + \lambda]\}}}{2({}^{r+1}c_2 - 3\lambda)} \quad (3)$$

The roots are real where $9\lambda^2(r-1)^2 \geq ({}^{r+1}c_2 - 3\lambda)\{r(b-1) - 3[(r-\lambda)^2 + \lambda(b-2r) + \lambda]$ Choose any real value for 'β', obtain the level α by choosing from $\alpha^2 = t\beta^2$. The resulting design D provides v-dimensional Second Order Rotatable Design in five levels.

THEOREM-2: A Second Order Rotatable Design with five levels (±α,±β, 0) can be constructed using if there exists Balanced Ternary Design with parameters V = v, B = $^{b}C_2$, R = r(b-1), K = 2k and π = (b-2)λ + r^2 then a SORD can be constructed with five levels such that where $\alpha^2 = t\beta^2$ or $t = \alpha^2 / \beta^2$.

Proof: Let N_{BXV} be the incidence matrix of a Balanced Ternary Design with parameters V = v, B = $^{b}C_2$, R = r(b-1), K = 2k and π = (b-2)λ + r^2. Each column of the incidence matrix has the elements 2's, 1's and 0's which are repeated $^{r}C_2$, r(b-r) and $^{b-r}C_2$ times respectively. Every pair of columns contains the pairs (2,2), (1,1) and ((2,1) or (1,2)) occurs $^{\lambda}C_2$, $[(r-\lambda)^2 + \lambda(b-2r+\lambda)]$ and 2λ(r-λ) times respectively Replace 2 with α and 1 with β. Associate each block with the levels ±1 by taking an appropriate fraction of factorials (say 2^{k_1}) for v factors. After augmenting n_0 central points, the resulting design has n = 2^{k_1} .$^{b}C_2$ + n_0 design points. Then we have,

$$2^{k_1}\,[\,{}^{r}C_2\,\alpha^4 + r(b-r)\beta^4) = 3.\,2^{k_1}\{{}^{\lambda}C_2\,\alpha^4 + [(r-\lambda)^2 + \lambda(b-2r+\lambda)]\beta^4 + 2\lambda(r-\lambda)\,\alpha^2\beta^2\} \quad (4)$$

From the rotatable condition, we obtains

$$({}^{r}C_2 - 3\,{}^{\lambda}C_2)\alpha^4 + \{r-(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]\beta^4 - 6\lambda(r-\lambda)\alpha^2\beta^2 = 0 \quad (5)$$

Let $t = \alpha^2/\beta^2$, then we can obtain

$$({}^{r}C_2 - 3\,{}^{\lambda}C_2)\,t^2 - 6\lambda(r-\lambda)t + \{r-(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]\} = 0 \quad (6)$$

by solving (6), we get $t = \frac{6\lambda\lambda(-\lambda)\sqrt{36\lambda^2(r-\lambda)^2 - 4({}^{r}c_2 - 3\,{}^{\lambda}c_2)\{r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]\}}}{2({}^{r}c_2 - 3^{\lambda}c_2)}$ for the existence of real valued of t, i.e.

$$36\lambda^2(r-\lambda)^2 \geq 4({}^{r}c_2 - 3\,{}^{\lambda}c_2)\{r(b-r) - 3[(r-\lambda)^2 + \lambda(b-2r+\lambda)]\} \quad (7)$$

for any real value of 'β', the value 'α' can be obtained using $\alpha^2 = t\beta^2$. The resulting design D provides a v-dimensional Second Order Rotatable Design in five levels.

EXAMPLE-1: Let N_1 be the incidence matrix of a Balanced Incomplete Block Design with parameters v = 4, b = 6, r = 3, k = 2 and λ = 1 and N be the derived incidence matrix of Balanced Ternary Design with parameters V = 4, B = 21, R = 21, K = 4 and $\pi = 17$. Then the resulting Second Order Rotatable Design in four factors is presented in Table I.

The real solution for ‘t’ is 4. Select value for β arbitrarily and accordingly α can be evaluated. $\alpha^2 = 4\beta^2$, when $\beta = \pm 1$ then $\alpha = \pm 4$.

TABLE I:CONSTRUCTION OF SECOND ORDER ROTATABLE DESIGN

BIBD(N_1)	BTD (.N)	SORD
1 1 0 0	2 2 0 0	$\pm\alpha$ $\pm\alpha$ 0 0
1 0 1 0	2 0 2 0	$\pm\alpha$ 0 $\pm\alpha$ 0
1 0 0 1	2 0 0 2	$\pm\alpha$ 0 0 $\pm\alpha$
0 1 1 0	0 2 2 0	0 $\pm\alpha$ $\pm\alpha$ 0
0 1 0 1	0 2 0 2	0 $\pm\alpha$ 0 $\pm\alpha$
0 0 1 1	0 0 2 2	0 0 $\pm\alpha$ $\pm\alpha$
	2 1 1 0	$\pm\alpha$ $\pm\beta$ $\pm\beta$ 0
	2 1 0 1	$\pm\alpha$ $\pm\beta$ 0 $\pm\beta$
	1 2 1 0	$\pm\beta$ $\pm\alpha$ $\pm\beta$ 0
	1 2 0 1	$\pm\beta$ $\pm\alpha$ 0 $\pm\beta$
	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$
	2 0 1 1	$\pm\alpha$ 0 $\pm\beta$ $\pm\beta$
	1 1 2 0	$\pm\beta$ $\pm\beta$ $\pm\alpha$ 0
	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$
	1 0 2 1	$\pm\beta$ 0 $\pm\alpha$ $\pm\beta$
	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$
	1 1 0 2	$\pm\beta$ $\pm\beta$ 0 $\pm\alpha$
	1 0 1 2	$\pm\beta$ 0 $\pm\beta$ $\pm\alpha$
	0 2 1 1	0 $\pm\alpha$ $\pm\beta$ $\pm\beta$
	0 1 2 1	0 $\pm\beta$ $\pm\alpha$ $\pm\beta$
	0 1 1 2	0 $\pm\beta$ $\pm\beta$ $\pm\alpha$

***EXAMPLE-2*:** Let N_1 be the incidence matrix of a Balanced Incomplete Block Design with parameters v = 4, b = 6, r = 3, k = 2 and $\lambda = 1$ and N be the derived incidence matrix of Balanced Ternary Design with parameters V = 4, B = 15, R = 15, K = 4 and $\pi = 13$. Then the resulting Second Order Rotatable Design in 4 factors is presented in Table II.

The real solution for ‘t’ is 4.4. Select value for β arbitrarily and accordingly α can be evaluated.

TABLE II: CONSTRUCTION OF SECOND ORDER ROTATABLE DESIGN

BIBD(N_1)	BTD (.N)	SORD
1 1 0 0	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$
1 0 1 0	2 1 1 0	$\pm\alpha$ $\pm\beta$ $\pm\beta$ ± 0
1 0 0 1	1 2 0 1	$\pm\beta$ $\pm\alpha$ ± 0 $\pm\beta$
0 1 1 0	2 1 0 1	$\pm\alpha$ $\pm\beta$ ± 0 $\pm\beta$
0 1 0 1	1 2 1 0	$\pm\beta$ $\pm\alpha$ $\pm\beta$ ± 0
0 0 1 1	1 0 2 1	$\pm\beta$ ± 0 $\pm\alpha$ $\pm\beta$
	0 1 1 2	± 0 $\pm\beta$ $\pm\beta$ $\pm\alpha$
	1 0 1 2	$\pm\beta$ ± 0 $\pm\beta$ $\pm\alpha$
	0 1 2 1	± 0 $\pm\beta$ $\pm\alpha$ $\pm\beta$
	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$
	2 0 1 1	$\pm\alpha$ ± 0 $\pm\beta$ $\pm\beta$
	1 1 2 0	$\pm\beta$ $\pm\beta$ $\pm\alpha$ ± 0
	1 1 0 2	$\pm\beta$ $\pm\beta$ ± 0 $\pm\alpha$
	0 2 1 1	± 0 $\pm\alpha$ $\pm\beta$ $\pm\beta$
	1 1 1 1	$\pm\beta$ $\pm\beta$ $\pm\beta$ $\pm\beta$

ACKNOWLEDGEMENT

The authors are thankful to the referee for their valuable comments in improving this manuscript.

FUNDING

Not applicable.

CONFLICT OF INTEREST

Not applicable.

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