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RESEARCH ARTICLE

The Arcsine Log-Logistic Distribution and Its Applications

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ABSTRACT

This article introduces Arcsine Log-logistic (AL-L) distribution which is a member of Arcsine-G family proposed by Rahman. The different properties of AL-L distribution have been discussed. The distribution of various order statistics are obtained. Maximum Likelihood Estimation (MLE) technique is used to estimate the model parameters. Finally, for testing the practicability of proposed model we have been used two different datasets.

Keywords: Arcsine-G family, Arcsine Log-logistic (AL-L), Log-logistic (L-L) distribution, Maximum Likelihood Estimation (MLE).

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1. Introduction

In probability theory and statistics, there are many situations in life testing experiments where observed datasets may not be fitted using known standard distributions. Generalization of such distributions by means of introducing new parameters have been successfully adopted in many applications. The logistic function was introduced in a series of three papers by Bacäer [1], who devised it as a model of population growth by adjusting the exponential growth model. There have too many generalization on it. Three types of generalized logistic distribution are proposed by Balakrishnan and Leung [2]. Nadarajah [3] proposed the skew logistic distribution. Berkson [4] used Logistic distribution instead of using normal distribution to solve many complex higher order real life phenomena.

The log-logistic distribution is very useful in a wide variety of applications, especially in the analysis of survival data used by Bennett [5], O'Quigley and Struthers [6], and Cox and Snell [7]. It has a non-monotonic hazard function that's makes it as a suitable way for modeling in real life data that applied in many fields like cancer survival data used by Ashkar *et al*. [8]. It is well appreciated in case of censored data usually common in reliability and life-testing experiments discussed by Tahir *et al*. [9]. There are so many generalization forms of this distribution are developed in many times called generalized Log-logistic distributions. The Beta log-logistic distribution was introduced by Lemonte [10]. The Kumaraswamy-log-logistic distribution was developed by De Santana *et al*. [11]. Inverse Log-logistic distribution for Extreme Wind Speed modeling was proposed by Chiodo *et al*. [12]. Shaw and Buckley [13] introduced transmuted family of distribution, to solve the problems related to the financial mathematics. Granzotto and Louzada [14] developed transmuted Log-logistic distribution. It can also be obtained by the equation proposed by Shaw and Buckley [15]. The Cubic transmuted family of distribution and hence Cubic transmuted Log-logistic distribution is introduced by Rahman [16]. Some applications of the log-logistic distribution are discussed in economy to model the wealth and income by Kleiber and Kotz [17] and in hydrology to model stream flow data by Ashkar and Mahdi [8]. In order to increase the applicability of this distribution, an extension will be made by using the family developed by Rahman [16] and defined as:

$$F(x) \quad = \quad \frac{2}{\pi} Arcsine[\sqrt{(G(x)}]; \quad x \in \mathbb{R} \tag{1}$$

where $G(x)$ indicates the (cdf) of baseline distribution for any random variable X.

So, for any (cdf) G(x) in (1), we may get several new members of this family. So using G(x) for Log-logistic random variable AL-L (a member of Arcsine-G family) is developed and detail study of it has been illustrated by the following several sections.

This article has been designed: Section II describes Arcsine Log-logistic (AL-L) distribution, the several properties of AL-L model are illustrated in section III. The definition and detail explanations of various order statistics of AL-L are given in section IV. Section V describes parameter estimation and inference procedure. Applications for two real datasets are shown in section VI for this AL-L model. The final section is about drawing inference for overall study.

2. Arcsine Log-Logistic Distribution

If X be a continuous Log-logistic random variable then the *cdf* of it is defined by

$$G(x) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}; \quad x \in (0, \infty), \tag{2}$$

where $\alpha > 0$, is scale and $\beta > 0$ is shape parameters respectively. Using (2) in (1), The *cdf* of the proposed AL-L distribution is as

$$\begin{aligned} F(x) &= \frac{2}{\pi} arcsine[\sqrt{(G(x)}] \\ &= \frac{2}{\pi} arcsine\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right]; \quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters respectively. The *pdf* of the proposed AL-L distribution is find out for the differentiation (3) with respect to x, and defined as

Definition: The density function of **Arcsine Log-logistic** distribution for X can be written as

$$f(x) = \frac{\beta\sqrt{\left(\frac{x}{\alpha}\right)^{\beta}}}{\pi x\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}$$

where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters respectively. Some different shapes of proposed AL-L model are shown for various combinations of the model parameters as Fig. 1:

Fig. 1, presents different behavior of parameters α and β for the proposed AL-L distribution and follows the basic properties of the distribution and density function when applied in the real-life phenomena.

3. Distributional Properties

This section describes different properties of the proposed AL-L distribution as below.

3.1. Moments

The moments of AL-L be:

$$\mu'_r = \alpha^r Sec\left[\frac{\pi r}{\beta}\right]; \quad \beta > 2r. \tag{4}$$

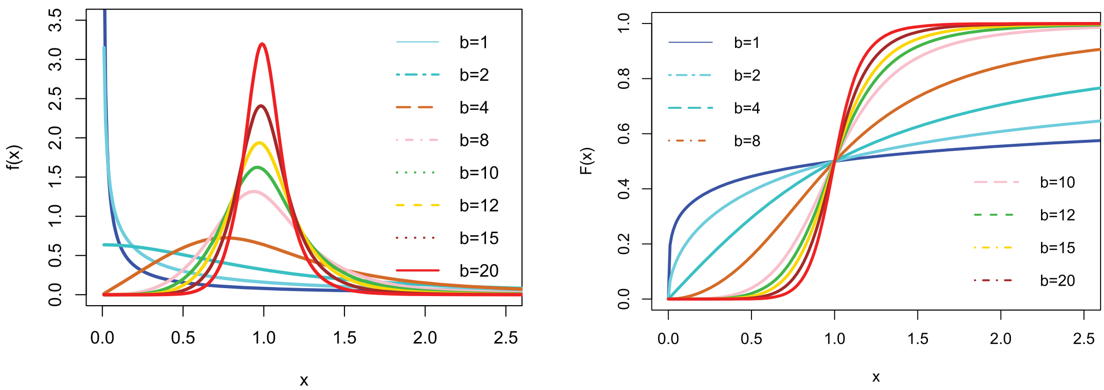

Fig. 1. The *pdf* (Left) and *cdf* (Right) for different combinations of α and β of AL-L distribution.

3.2. Mean, Variance, Skewness and Kurtosis

By putting $r = 1, 2, 3, 4$ in (4), we can get first four moments respectively and higher order moments for larger values as well and hence $E(X) = \mu = \mu_1' = \alpha\ Sec\left[\frac{\pi}{\beta}\right]$; $\beta > 2$. and $\sigma^2 = \mu_2' - (\mu_1')^2 = \alpha^2\ \tan^2\left(\frac{\pi}{\beta}\right)\ \sec\left(\frac{2\pi}{\beta}\right)$; $\beta > 4$. In the case of mean $\beta > 2$, and in variance $\beta > 4$, It can be easily shown that the mean and variance chart for the different values of parameters α and β as in Table I and Fig. 2.

According to Table I, the mean are decreasing into horizontal axes with respect to change of the parameters α and β and also same cases exist for the variance.

It can also be obtained $Skewness = \gamma_1 = \frac{\mu_3^2}{\mu_2^3} = \cos^3\left(\frac{2\pi}{\beta}\right)\cot^6\left(\frac{\pi}{\beta}\right)\left[\frac{1}{2}\sec^3\left(\frac{\pi}{\beta}\right)\left(1 - 3\sec\left(\frac{2\pi}{\beta}\right)\right) + \sec\left(\frac{3\pi}{\beta}\right)\right]^2$, $\beta > 6$. and $Kurtosis = \gamma_2 = \frac{\mu_4}{(\mu_2)^2}, = \left[\frac{\cos^2\left(\frac{2\pi}{\beta}\right)\cot^4\left(\frac{\pi}{\beta}\right)}{\alpha^4}\right] \times \left[\left(-3\alpha^4\sec^4\left(\frac{\pi}{\beta}\right) + 6\alpha^4\sec^2\left(\frac{\pi}{\beta}\right)\sec\left(\frac{2\pi}{\beta}\right)\right) - \left(4\alpha^4\sec\left(\frac{\pi}{\beta}\right)\sec\left(\frac{3\pi}{\beta}\right) + \alpha^4\sec\left(\frac{4\pi}{\beta}\right)\right)\right]$; $\beta > 8$.

3.3. Moments Generating Function (MGF)

The MGF of AL-L model can be explained as below.

Theorem: If X be a continuous in type random variable of AL-L model, then the MGF $M_X(t)$ is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \cdot \alpha^r Sec\left[\frac{\pi r}{\beta}\right].$$

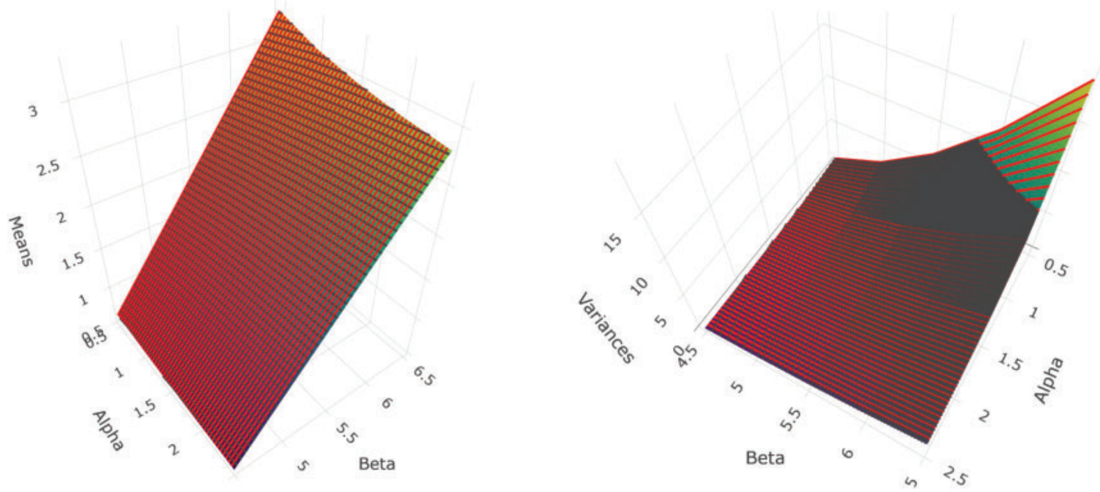

Fig. 2. The plot of mean (Left) and variance (Right) for different values of α and β of the proposed AL-L distribution.

TABLE I: The Mean and Variance (in Parentheses) of the AL-L Distribution

	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$
β =4.5	0.653	0.618	0.594	0.577	0.565
	(0.763)	(0.337)	(0.204)	(0.141)	(0.105)
$\beta = 5$	1.305	1.236	1.189	1.155	1.129
	(3.052)	(1.348)	(0.815)	(0.562)	(0.419)
$\beta = 5.5$	1.958	1.854	1.783	1.732	1.694
	(6.868)	(3.034)	(1.833)	(1.266)	(0.942)
$\beta = 6$	2.611	2.472	2.377	2.309	2.259
	(12.209)	(5.393)	(3.258)	(2.250)	(1.675)
$\beta = 6.5$	3.264	3.090	2.972	2.887	2.823
	(19.077)	(8.427)	(5.091)	(3.515)	(2.618)

where $t \in \mathbb{R}$.

proof: The MGF is defined by

$$M_X(t) = E[e^{tX}]$$
$$= \int_0^\infty e^{tx} f(x)\mathrm{d}x,$$

where $f(x)$ is *pdf* of proposed AL-L. By series expansion of e^{tx} presented in [18], we get

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} x^r f(x)\mathrm{d}t$$
$$= \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r). \tag{5}$$

From (4) putting $E(X^r)$ in (5), finally we get $M_X(t)$.

3.4. Characteristic Function (CF)

The CF always exists when treated as a function of a real-valued argument. It is any real-valued random variable completely defines its probability distribution. The CF of AL-L model is described as below.

Theorem: If X follows the AL-L model, the CF, $\phi_X(t)$ is

$$\phi_X(t) = \sum_{r=0}^\infty \frac{(\mathrm{i}t)^r}{r!} \cdot \alpha^r Sec\left[\frac{\pi r}{\beta}\right],$$

where $t \in \mathbb{R}$ and $i = \sqrt{-1}$ is an imaginary unit.

proof: It is easy to prove like MGF.

3.5. Reliability and Hazard Function

The function of reliability for the proposed AL-L distribution denoted by R(t) and defined by

$$R(t) = 1 - F(t)$$
$$= 1 - \frac{2}{\pi} Arcsine\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right]$$
$$= 1 - \frac{2}{\pi} Arcsine\left[\frac{1}{\sqrt{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right].$$

and the hazard function is the ratio of the probability distribution function to the reliability function and is given by

$$h(t) = \frac{f(t)}{R(t)}$$
$$= \frac{f(t)}{1-F(t)}$$
$$= \frac{\beta\sqrt{\dfrac{\left(\frac{x}{\alpha}\right)^\beta}{\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^2}}}{2x\cos^{-1}\left(\sqrt{\dfrac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right)}.$$

Fig. 3 shows some shapes of reliability and hazard function of proposed AL-L model that indicate the ability to draw the complexity of the real-life datasets in reliability and hazard analysis.

3.6. *Quantile Function and Median*

The quantile function of AL-L distribution is denoted by x_q, and found by the solution of $F(x) = q$ as below.

$$x_q = \frac{2}{\pi} arcsine \left[\sqrt{\frac{1}{1 + \left(\frac{x}{\alpha}\right)^{-\beta}}}, \right]$$

which can be further obtained as

$$x_q = \alpha \left(-\frac{\sin^2\left(\frac{\pi q}{2}\right)}{\sin^2\left(\frac{\pi q}{2}\right) - 1} \right)^{1/\beta}, \tag{6}$$

By using (6), one can easily obtain the first quartile (Q_1), second quartile (Q_2) or median and third quartile (Q_3) by setting $q = 0.25,\ 0.50$, and 0.75 respectively.

3.7. *Random Number Generation*

The random number from the proposed AL-L distribution is generated by using

$$x = \frac{2}{\pi} arcsine \left[\sqrt{\frac{1}{1 + \left(\frac{x}{\alpha}\right)^{-\beta}}} \right].$$

where, $u \sim U(0, 1)$, and also it can be determined as

$$X = \alpha \left(\frac{\sin^2\left(\frac{\pi u}{2}\right)}{1 - \sin^2\left(\frac{\pi u}{2}\right)} \right)^{1/\beta}, \tag{7}$$

where $0 < u < 1$.

Again, one can use (7) for the random number generation for further analysis.

4. Order Statistics

The probability density function of *rth* order statistic for AL-L distribution is given by following

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x)$$

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!} \left[\frac{\beta\sqrt{(\frac{x}{\alpha})^{\beta}}}{\pi x \left(1 + \left(\frac{x}{\alpha}\right)^{\beta}\right)} \right]$$

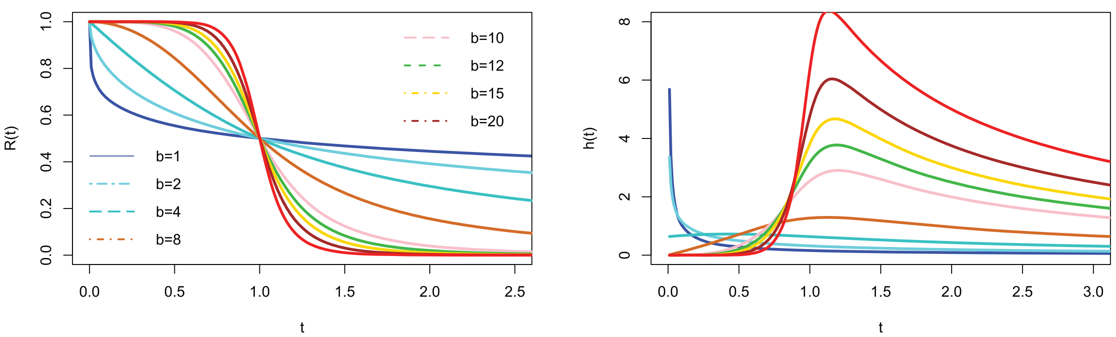

Fig. 3. The function $R(t)$ (Left) and $h(t)$ (Right) for different α and β respectively of AL-L distribution.

$$\times\left[\frac{2}{\pi}Arcsine\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right]\right]^{r-1}$$

$$\times\left[1-\frac{2}{\pi}Arcsine\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}}\right]\right]^{n-r}, \tag{8}$$

for $r = 1, 2, \cdots, n$. Now putting $r = 1$, in (8), we get the function of first order statistic and is defined as

$$f_{X_{1:n}}(x) = \left[\frac{n!\,\beta 2^{n-1}\pi^{-n}\sqrt{\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}}}\,ArcCos\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-}\beta}}\right]^{n-1}}{x(n-1)!}\right]$$

$$= \left[\frac{n!\,\beta 2^{n-1}\sqrt{\left(\frac{x}{\alpha}\right)^{\beta}}\,ArcCos\left[\sqrt{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-}\beta}}\right]^{n-1}}{\pi^{n}x(n-1)!\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right],$$

and also putting $r = n$, in (8), we get the last order statistic and is defined by

$$f_{X_{n:n}}(x) = \left[\frac{n!\,\beta\sqrt{\left(\frac{x}{\alpha}\right)^{\beta}}\left(\pi-2ArcCos\left[\sqrt{\frac{1}{1+(\frac{x}{\alpha})^{-}\beta}}\right]^{n-1}\right)}{x\pi^{n}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\Gamma(n)}\right].$$

The kth order moment of AL-L model is found by

$$E(X_{r:n}^{k}) = \int_{0}^{\infty} x_r^k \cdot f_{X_{r:n}}(x)\cdot \mathrm{d}x.$$

5. Model Parameter Estimation and Inference

The maximum likelihood estimation technique has been used to estimate the parameter of proposed AL-L model. Suppose the likelihood function of $x_1, x_2, \cdots, x_n$ random sample of AL-L distribution is defined as

$$L = \prod_{i=1}^{n}\left[\frac{\beta\sqrt{\left(\frac{x}{\alpha}\right)^{\beta}}}{\pi x\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right],$$

and the equivalent function $l = ln(L)$ be

$$l = n\log[\beta] + \sum_{i=1}^{n}\log\left[\sqrt{\left(\frac{x}{\alpha}\right)^{\beta}}\right]$$

$$-\,n\log[\pi] - \sum_{i=1}^{n}\log[x_i] - \sum_{i=1}^{n}\log\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]. \tag{9}$$

TABLE II: Summary Statistics of Selected Datasets

Data set	Min.	Q1	Median	Mean	Q3	Max
Carbon fiber	0.390	1.804	2.700	2.621	3.220	5.560
Aircraft windshield	0.040	1.848	2.367	2.580	3.400	4.694

The parameter α and β are estimated by maximizing the (9) and hence we obtain

$$\frac{\partial l}{\partial \alpha} = \frac{\beta n x \left(\frac{x}{\alpha}\right)^{\beta-1}}{\alpha^2\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)} - \frac{n\beta}{2\alpha},$$

and

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \frac{nlog\left[\frac{x}{\alpha}\right]}{2} - \frac{n\left(\frac{x}{\alpha}\right)^{\beta} log\left[\frac{x}{\alpha}\right]}{1+\left(\frac{x}{\alpha}\right)^{\beta}}.$$

by solving $\frac{\partial l}{\partial \alpha} = 0$, and $\frac{\partial l}{\partial \beta} = 0$ we get $\hat{\Theta} = \left(\hat{\alpha}, \hat{\beta}\right)'$ of $\Theta = (\alpha, \beta)'$. Also the MLE's are asymptotically distributed for $n \to \infty$, as like as [19], [20]

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & \hat{V}_{22}\end{pmatrix}\right]$$

By the inverting Hessian matrix we can get the asymptotic variance-covariance matrix V, of $\hat{\alpha}, \hat{\beta}$ given in "Appendix". For α and β, the two sided $100(1-\alpha)$ approximate confidence intervals are

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}, \text{ and}$$

$$\hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}.$$

where αth percentile of the standard normal distribution is Z_{α}.

6. Real-Life Applications

6.1. Data Sources

Two types of real life datasets are used from different sources such as i) Breaking strength of carbon fibers data of 50 mm length (GPa) is obtained from Ashkar and Mahdi [8]. This dataset is also used in Cubic rank transmuted distributions by Granzotto [21], and ii) Aircraft windshield failures Data (thousands of hour) represents the remission times (in months) of a random sample. This datasets is also used in so many research as Mixture models for analyzing product reliability data by using Weibull mixture mode by Sea [22], and El-Bassiouny *et al*. [23] in exponential lomax distribution and also used it in kumaraswamy-G family by Afifya *et al*. [24].

Table II describes summary statistics of these datasets that indicates the asymmetric distribution.

For testing the applicability of AL-L model, we have been considered existed Log-logistic (L-L) and Cubic Transmuted Log-logistic (CTL-L) distribution and we obtain the outcomes shown as Table III.

From Fig. 4, it is proved that the proposed AL-L distribution is well fitted compare to other selected models for selected datasets. Again, the different values of Log-likelihood, Akaike's information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC) have been calculated for selected models under selected datasets mathematically that are shown in Table IV.

The results obtained in Table IV illustrate that the new AL-L distribution provides higher value of log-likelihood and Lower value of AIC, AICc, and BIC than other two existing distributions, So the proposed AL-L distribution is better fitted for the under studied real datasets.

TABLE III: Diferent MLE's and Their Corresponding *SE*'s Values for Different Model Parameters

Datasets	Distribution	Parameters	Estimate	SE
Carbon fiber	AL-L	α	7.499	0.959
		β	2.750	0.168
	CTL-L	λ_1	1.00	–
		λ_2	1.00	–
		α	3.5341	–
		β	2.9231	–
	L-L	α	2.4983	0.1054
		β	4.1176	0.3440
Aircraft windshield	AL-L	α	7.499	1.015
		β	2.750	0.179
	CTL-L	α	3.504	–
		β	2.568	0.291
		λ_1	1.00	–
		λ_2	0.999	1.066
	L-L	α	2.412	0.133
		β	3.260	0.293

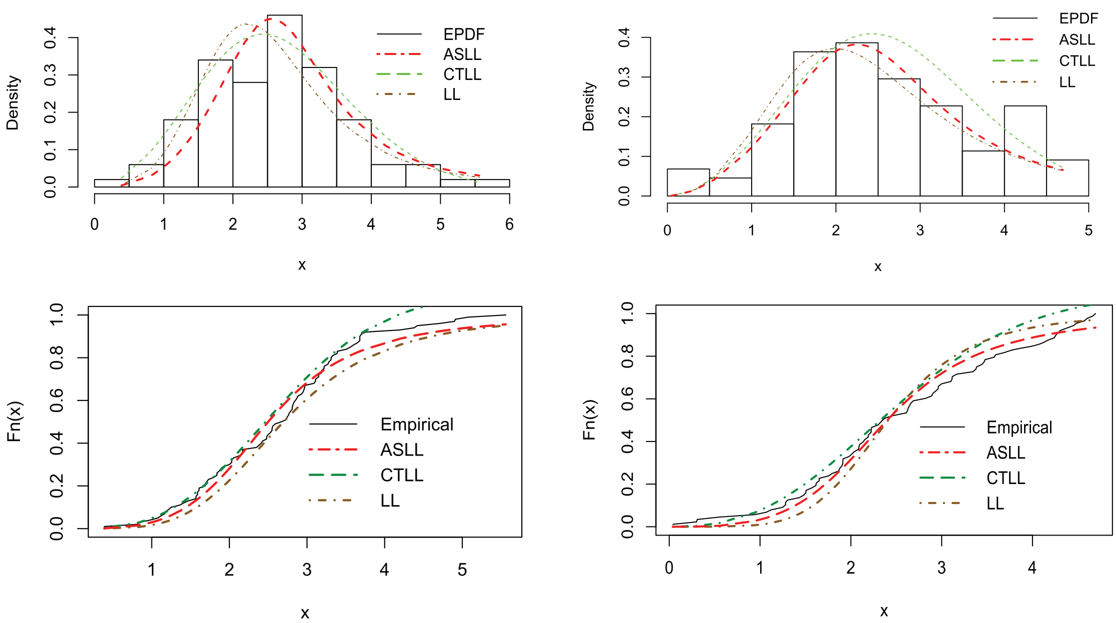

Fig. 4. Estimated PDF (Uper panel) and CDF (Lower panel) along with Empirical PDF and CDF for under studied datasets.

TABLE IV: Different Values of Some Model Selection Criteria for Different Models

Datasets	Distribution	Log-likelihood	AIC	AICc	BIC
Carbon fiber	AL-L	−126.5819	257.1639	257.2876	262.3742
	CTL-L	−132.8568	273.7137	274.1347	284.1343
	L-L	−146.2794	296.559	296.682	301.769
Aircraft windshield	AL-L	−114.020	232.041	232.183	236.996
	CTL-L	−129.380	266.761	267.243	276.671
	L-L	−146.138	296.277	296.418	301.232

7. Summary and Conclusions

In this article, a new Arcsine Log-logistic (AL-L) distribution has been proposed, which is flexible enough to handle the complex (bi-modal) data. The distributional properties including moments, generating functions, quantile function, reliability function, hazard rate function, and the distributions of several ordered values for new AL-L distribution has been discussed. For estimating the parameters besides standard errors, estimation technique of maximum likelihood has been applied. The brief discussion about the inferential procedure of the model parameters has also been discussed. For assessing the practicality of the proposed AL-L distribution, two real-life datasets, including environmental and life-time datasets have been considered. The estimated pdf and cdf plots over empirical pdf and cdf have been drawn that shows better fits in favor of the proposed model. On the basis of computed values of the selection criterion's, it has been observed that the two parameters proposed AL-L distribution shown better fits as compared with the other competing distributions like three parameters Cubic Transmuted Log-logistic (CTL-L) distribution and two parameters Log-logistic (L-L) base distribution used in this research, for modeling of various data.

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Conflict of Interest

The authors declare that they do not have any conflict of interest.

Appendix

The Hessian Matrix for the Proposed AL-L Distribution

The Hessian matrix is given as

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

and the matrix (variance-covariance) V is

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}^{-1}.$$

With the elements

$$\begin{aligned} H_{11} &= -\frac{\delta^2 l}{\delta\alpha^2} = n\left[-\frac{\beta^2 x^2 \left(\frac{x}{\alpha}\right)^{2\beta-2}}{\alpha^4\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)^2} + \frac{(\beta-1)\beta x^2\left(\frac{x}{\alpha}\right)^{\beta-2}}{\alpha^4\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)}\right] \\ &\quad + \left[\frac{2n\beta x\left(\frac{x}{\alpha}\right)^{\beta-1}}{\alpha^3\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)} - \frac{\beta n}{2\alpha^2}\right], \end{aligned}$$

$$\begin{aligned} H_{12} &= -\frac{\delta^2 l}{\delta\alpha\,\delta\beta} = \frac{n}{2\alpha} - \frac{nx\left(\frac{x}{\alpha}\right)^{\beta-1}}{\alpha^2\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)} \\ &\quad - \frac{\beta nx\left(\frac{x}{\alpha}\right)^{\beta-1}\log\left(\frac{x}{\alpha}\right)}{\alpha^2\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)} + \frac{\beta nx\left(\frac{x}{\alpha}\right)^{2\beta-1}\log\left(\frac{x}{\alpha}\right)}{\alpha^2\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)^2}, \end{aligned}$$

$$\begin{aligned} H_{21} &= -\frac{\delta^2 l}{\delta\beta\,\delta\alpha} = \frac{n}{2\alpha} - \frac{\beta nx\left(\frac{x}{\alpha}\right)^{\beta-1}\log\left(\frac{x}{\alpha}\right)}{\alpha^2\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)} \\ &\quad + \frac{\beta nx\left(\frac{x}{\alpha}\right)^{2\beta-1}\log\left(\frac{x}{\alpha}\right)}{\alpha^2\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)^2} - \frac{n\left(\frac{x}{\alpha}\right)^{\beta}}{\alpha\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)} \end{aligned}$$

$$H_{22} \quad = \quad -\frac{\delta^2 l}{\delta\beta^2} = \frac{n}{\beta^2} + n\left(\frac{\left(\frac{x}{\alpha}\right)^{\beta}\log^2\left(\frac{x}{\alpha}\right)}{\left(\frac{x}{\alpha}\right)^{\beta}+1} - \frac{\left(\frac{x}{\alpha}\right)^{2\beta}\log^2\left(\frac{x}{\alpha}\right)}{\left(\left(\frac{x}{\alpha}\right)^{\beta}+1\right)^2}\right)$$

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