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RESEARCH ARTICLE

Introduction to the Thukral-Determinantal Formula for Accelerating Convergence of Sequence

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ABSTRACT

There are two objectives for this paper. Firstly, we shall introduce the Thukral-determinantal formula, and secondly, we shall demonstrate the similarities between the well-established algorithms, namely the Aitkin Δ^2 algorithm and the Durbin sequence transformation. In fact, we have found that the solution of the Thukral-determinantal formula is equivalent to the Thukral-sequence transformation formula.

Keywords: Aitkin Δ2 algorithm, Brezinski-Durbin-Redivo-Zaglia sequence transformation, Durbin sequence transformation, Thukral-determinantal formula.

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1. Introduction

In almost all areas of numerical mathematics, there are convergence problems. In such cases, either the convergence is slow or even divergence is observed and accordingly many techniques for accelerating convergence have been devised. For example; Aitkin Δ^2 algorithm, ε-algorithm, θ-algorithm, Lubkin transformation, Levin-type transformation, and many others may be found in [1], [2]. Whereas Durbin's sequence transformation and Brezinski-Durbin-Redivo-Zaglia's sequence transformation may be found in [3], [4]. It is well-established that the Durbin sequence transformation cannot be expressed as a ratio of two determinants [5], [6], and it has been a long quest to find this representation. Hence, the prime motive for the introduction of the Thukral-determinantal formula was to establish that the Durbin sequence transformation can be represented as a ratio of two determinants. Furthermore, we shall verify that the Thukral-determinantal formula produces an equivalent equation to the Thukral-sequence transformation, and there is a particular connection between the Brezinski-Durbin-Redivo-Zaglia's sequence transformation.

The structure of this paper is as follows: In Section 2, we state the essential definitions relevant to the present work. In Section 3, we shall define the Thukral-determinantal formula, and in process we will demonstrate the similarities between the Aitkin Δ^2 algorithm, the Durbin sequence transformation, and the Thukral-sequence transformation. In order to construct the new determinantal formula, the following proposition is essential.

2. Preliminaries

In order to construct an iterative scheme, the following assumptions are essential and are given in [1], [2].

Let us assume that

$$s = \sum_{i=0}^{\infty} c_i, \tag{1}$$

is a slowly convergent or divergent sequence, whose elements s_n are the partial sum of an infinite series, give as

$$s_n = \sum_{i=0}^{n} c_i. \tag{2}$$

The basic assumption of all sequence transformation is that a sequence element s_n can be for all indices $n \geq 0$ to be partitioned into a limit s a truncation error e_n according to

$$s_n = s + e_n. \tag{3}$$

The conventional approach of evaluating an infinite series consists of adding up so many terms that the error e_n ultimately becomes zero. Unfortunately, this is not always possible because of obvious practical limitations. Moreover, adding up further terms does not work in the case of a divergent series. Therefore, the development of new techniques plays an important role.

Furthermore, let us assume the two sequences $|s_n|$ and $\left|s_n^{'}\right|$ both converge to the same limit s. Another condition of the sequence $\left|s_n^{'}\right|$ converges more rapidly than $|s_n|$ then

$$\lim_{n\to\infty} \frac{s_n^{'} - s}{s_n - s} = 0. \tag{4}$$

3. Construction of the Thukral-Determinant Formula

In this section, we shall define the Thukral-determinantal formula for accelerating the convergence of sequences. In the process, we shall demonstrate that the Thukral-determinantal formula is in fact equivalent to the Thukral-sequence transformation formula and show a connection with the Brezinski-Durbin-Redivo-Zaglia [3], [4]. Hence, we shall show the similarities for the first five cases and then define the Thukral-determinantal formula as a general form. We begin with the Brezinski-Durbin-Redivo-Zaglia formula,

$$BDRZ_n = \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} s_{n+i} \Delta s_{n+k-i}}{\sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta s_{n+k-i}}, \tag{5}$$

where $n,\ k \in \aleph$, Δ operates, now and in sequel, on the variable n. For example,

$$\Delta s_n = (s_{n+1} - s_n)\,.$$

In order to obtain the identical results between the Thukral-determinantal formula and the Thukral-sequence transformation we will modify the Brezinski-Durbin-Redivo-Zaglia formula, and define a new version, namely Thukral-sequence transformation as

$$t_n = \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} s_{n+i+1} \Delta s_{n+k-i}}{\sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta s_{n+k-i}}. \tag{6}$$

We shall demonstrate the similarities between the Thukral-determinantal formula and Thukral-sequence transformation for several values of k.

For case $k = 1$:

$$T_1 = (-1)^1 \frac{\begin{vmatrix} \dfrac{1}{\Delta s_n} & \dfrac{1}{\Delta s_{n+1}} \\ s_{n+1} & s_n \end{vmatrix}}{\begin{vmatrix} \dfrac{1}{\Delta s_{n+1}} & \dfrac{1}{\Delta s_n} \\ 1 & 1 \end{vmatrix}} = \frac{(s_n \Delta s_{n+1} - s_{n+1} \Delta s_n)\,(\Delta s_{n+1} \Delta s_n)^{-1}}{(\Delta s_n - \Delta s_{n+1})\,(\Delta s_{n+1} \Delta s_n)^{-1}}. \tag{7}$$

By cancelling common factors and simplifying (7), we get

$$T_1 = \frac{\left(s_{n+2} s_n - s_{n+1}^2\right)}{(s_{n+2} - 2s_{n+1} + s_n)}. \tag{8}$$

Using the Thukral-sequence transformation Formula (6) for case $k = 1$, we obtain

$$t_1 = \frac{\sum_{i=0}^{1} (-1)^i \binom{1}{i} s_{n+i+1} \Delta s_{n+1-i}}{\sum_{i=0}^{1} (-1)^i \binom{1}{i} \Delta s_{n+1-i}} = \frac{s_n \Delta s_{n+1} - s_{n+1} \Delta s_n}{\Delta s_{n+1} - \Delta s_n}. \tag{9}$$

Expanding and simplifying (9), we obtain

$$t_1 = \frac{\left(s_{n+2} s_n - s_{n+1}^2\right)}{\left(s_{n+2} - 2s_{n+1} + s_n\right)}. \tag{10}$$

The Aitkin Δ^2 algorithm is a well-known technique for accelerating the convergence of sequences and in fact, there are many versions of the algorithm [1]–[3], [7], the Eq. (10) is one of many versions of the Aitkin Δ^2 algorithm. Therefore, it is apparent that (8) and (10) are equivalent and is one of the versions of the Aitken process.

We progress to $k = 2$:

$$T_2 = (-1)^2 \frac{\begin{vmatrix} \frac{1}{\Delta s_n} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_{n+2}} \\ \frac{n}{\Delta s_n} & \frac{n+1}{\Delta s_{n+1}} & \frac{n+2}{\Delta s_{n+2}} \\ s_{n+2} & s_{n+1} & s_n \end{vmatrix}}{\begin{vmatrix} \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_n} \\ \frac{n}{\Delta s_{n+2}} & \frac{n+1}{\Delta s_{n+1}} & \frac{n+2}{\Delta s_n} \\ 1 & 1 & 1 \end{vmatrix}} = \frac{\left(s_n \Delta s_{n+2} - 2s_{n+1} \Delta s_{n+1} + s_{n+2} \Delta s_n\right) \left(\Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}{\left(\Delta s_n - 2\Delta s_{n+1} + \Delta s_{n+2}\right) \left(\Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}. \tag{11}$$

Expanding the numerator and denominator of (11), we get

$$T_2 = \frac{\left(s_n \Delta s_{n+2} - 2s_{n+1} \Delta s_{n+1} + s_{n+2} \Delta s_n\right) \left(\Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}{\left(\Delta s_n - 2\Delta s_{n+1} + \Delta s_{n+2}\right) \left(\Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}. \tag{12}$$

Again, we cancel the common factors and simplify (12), which yields

$$T_2 = \frac{\left(s_n \Delta s_{n+2} - 2s_{n+1} \Delta s_{n+1} + s_{n+3} \Delta s_n\right)}{\left(\Delta s_n - 2\Delta s_{n+1} + \Delta s_{n+2}\right)}. \tag{13}$$

The second scheme of the Thukral-sequence transformation Formula (6) is given as

$$t_2 = \frac{\sum_{i=0}^{2} (-1)^i \binom{2}{i} s_{n+i+1} \Delta s_{n+2-i}}{\sum_{i=0}^{2} (-1)^i \binom{2}{i} \Delta s_{n+2-i}}. \tag{14}$$

The Eq. (14) is simplified as

$$t_2 = \frac{\left(s_n \Delta s_{n+2} - 2s_{n+1} \Delta s_{n+1} + s_{n+2} \Delta s_n\right)}{\left(\Delta s_n - 2\Delta s_{n+1} + \Delta s_{n+2}\right)}. \tag{15}$$

Here also we find that the expressions (13) and (15) are identical.

For the purpose and motivation of this paper, we shall demonstrate similarities between the Thukral-determinantal formula and the Durbin sequence transformation for the next case.

When $k = 3$, the Thukral-determinantal formula is given as

$$T_3 = (-1)^3 \frac{\begin{vmatrix} \frac{1}{\Delta s_n} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+3}} \\ \frac{n}{\Delta s_n} & \frac{n+1}{\Delta s_{n+1}} & \frac{n+2}{\Delta s_{n+2}} & \frac{n+3}{\Delta s_{n+3}} \\ \frac{n^2}{\Delta s_n} & \frac{(n+1)^2}{\Delta s_{n+1}} & \frac{(n+2)^2}{\Delta s_{n+2}} & \frac{(n+3)^2}{\Delta s_{n+3}} \\ s_{n+3} & s_{n+2} & s_{n+1} & s_n \end{vmatrix}}{\begin{vmatrix} \frac{1}{\Delta s_{n+3}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_n} \\ \frac{n}{\Delta s_{n+3}} & \frac{n+1}{\Delta s_{n+2}} & \frac{n+2}{\Delta s_{n+1}} & \frac{n+3}{\Delta s_n} \\ \frac{n^2}{\Delta s_{n+3}} & \frac{(n+1)^2}{\Delta s_{n+2}} & \frac{(n+2)^2}{\Delta s_{n+1}} & \frac{(n+3)^2}{\Delta s_n} \\ 1 & 1 & 1 & 1 \end{vmatrix}} \tag{16}$$

Evaluating the above equation, we get

$$T_3 = \frac{2\left(s_n \Delta s_{n+3} + 3 s_{n+1} \Delta s_{n+2} + 3 s_{n+2} \Delta s_{n+1} - s_{n+3} \Delta s_n\right)\left(\Delta s_{n+3} \Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}{2\left(\Delta s_n - 3\Delta s_{n+1} + 3\Delta s_{n+2} - \Delta s_{n+3}\right)\left(\Delta s_{n+3} \Delta s_{n+2} \Delta s_{n+1} \Delta s_n\right)^{-1}}. \tag{17}$$

As before, we cancel the common factors of (17) and obtain

$$T_3 = \frac{\left(-s_n \Delta s_{n+3} + 3 s_{n+1} \Delta s_{n+2} - 3 s_{n+2} \Delta s_{n+1} + s_{n+3} \Delta s_n\right)}{\left(\Delta s_n - 3\Delta s_{n+1} + 3\Delta s_{n+2} - \Delta s_{n+3}\right)}. \tag{18}$$

We expand (18) and show the similarities between the Thukral-determinantal formula and the Durbin sequence transformation, thus becomes

$$T_3 = \frac{\left(-s_n\left(s_{n+4} - s_{n+3}\right) - 3 s_{n+1}\left(s_{n+3} - s_{n+2}\right) + 3 s_{n+2}\left(s_{n+2} - s_{n+1}\right) - s_{n+3}\left(s_{n+1} - s_n\right)\right)}{\left(-\left(s_{n+1} - s_n\right) + 3\left(s_{n+2} - s_{n+1}\right) - 3\left(s_{n+3} - s_{n+2}\right) + \left(s_{n+4} - s_{n+3}\right)\right)}. \tag{19}$$

By cancelling similar terms and simplifying (19), we get

$$T_3 = \frac{\left(3 s_{n+2}^2 - 4 s_{n+1} s_{n+3} + s_n s_{n+4}\right)}{\left(6 s_{n+2} - 4\left(s_{n+1} + s_{n+3}\right) + \left(s_n + s_{n+4}\right)\right)} \tag{20}$$

Which is identical to the Durbin sequence transformation [1], [3], [5]. Moreover, from the Thukral-sequence transformation Formula (6), for $k = 3$, yields

$$t_3 = \frac{\sum_{i=0}^{3} (-1)^i \binom{3}{i} s_{n+i} \Delta s_{n+3-i}}{\sum_{i=0}^{3} (-1)^i \binom{3}{i} \Delta s_{n+3-i}}, \tag{21}$$

$$t_3 = \frac{\left(s_n \Delta s_{n+3} - 3 s_{n+1} \Delta s_{n+2} + 3 s_{n+2} \Delta s_{n+1} - s_{n+3} \Delta s_n\right)}{\left(\Delta s_{n+3} - 3\Delta s_{n+2} + 3\Delta s_{n+1} - \Delta s_n\right)}. \tag{22}$$

Expanding and simplifying (22), we obtain

$$t_3 = \frac{\left(3 s_{n+2}^2 - 4 s_{n+1} s_{n+3} + 3 s_{n+2} s_{n+4} + s_n s_{n+4}\right)}{\left(6 s_{n+4} - 4\left(s_{n+1} - s_{n+1}\right) + 3\left(s_{n+3} - s_{n+2}\right)\right)}. \tag{23}$$

Consequently, for $k = 3$ we have demonstrated that the Thukral-determinantal formula, the Durbin sequence transformation and the Thukral-sequence transformation formula produce identical results.

The Thukral-determinantal formula for $k = 4$:

$$T_4 = (-1)^4 \frac{\begin{vmatrix} \frac{1}{\Delta s_n} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+3}} & \frac{1}{\Delta s_{n+4}} \\ \frac{n}{\Delta s_n} & \frac{n+1}{\Delta s_{n+1}} & \frac{n+2}{\Delta s_{n+2}} & \frac{n+3}{\Delta s_{n+3}} & \frac{n+4}{\Delta s_{n+4}} \\ \frac{n^2}{\Delta s_n} & \frac{(n+1)^2}{\Delta s_{n+1}} & \frac{(n+2)^2}{\Delta s_{n+2}} & \frac{(n+3)^2}{\Delta s_{n+3}} & \frac{(n+4)^2}{\Delta s_{n+4}} \\ \frac{n^3}{\Delta s_n} & \frac{(n+1)^3}{\Delta s_{n+1}} & \frac{(n+2)^3}{\Delta s_{n+2}} & \frac{(n+3)^3}{\Delta s_{n+3}} & \frac{(n+4)^3}{\Delta s_{n+4}} \\ s_{n+4} & s_{n+3} & s_{n+2} & s_{n+1} & s_n \end{vmatrix}}{\begin{vmatrix} \frac{1}{\Delta s_{n+4}} & \frac{1}{\Delta s_{n+3}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_n} \\ \frac{n}{\Delta s_{n+4}} & \frac{n+1}{\Delta s_{n+3}} & \frac{n+2}{\Delta s_{n+2}} & \frac{n+3}{\Delta s_{n+1}} & \frac{n+4}{\Delta s_n} \\ \frac{n^2}{\Delta s_{n+4}} & \frac{(n+1)^2}{\Delta s_{n+3}} & \frac{(n+2)^2}{\Delta s_{n+2}} & \frac{(n+3)^2}{\Delta s_{n+1}} & \frac{(n+4)^2}{\Delta s_n} \\ \frac{n^3}{\Delta s_{n+4}} & \frac{(n+1)^3}{\Delta s_{n+3}} & \frac{(n+2)^3}{\Delta s_{n+2}} & \frac{(n+3)^3}{\Delta s_{n+1}} & \frac{(n+4)^3}{\Delta s_n} \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}}. \tag{24}$$

Expanding (24), we obtain

$$T_4 = \frac{12\left(s_n\Delta s_{n+4} - 4s_{n+1}\Delta s_{n+3} + 6s_{n+2}\Delta s_{n+2} - 4s_{n+3}\Delta s_{n+1+}s_{n+4}\Delta s_n\right)\left(\Delta s_{n+4}\Delta s_{n+3}\Delta s_{n+2}\Delta s_{n+1}\Delta s_n\right)^{-1}}{12\left(\Delta s_n - 4\Delta s_{n+1} + 6\Delta s_{n+2} - 4\Delta s_{n+3} + \Delta s_{n+4}\right)\left(\Delta s_{n+4}\Delta s_{n+3}\Delta s_{n+2}\Delta s_{n+1}\Delta s_n\right)^{-1}}. \tag{25}$$

Eliminating the common factors and simplifying (25), we get

$$T_4 = \frac{\left(s_n\Delta s_{n+4} - 4s_{n+1}\Delta s_{n+3} + 6s_{n+2}\Delta s_{n+2} - 4s_{n+3}\Delta s_{n+1+}s_{n+4}\Delta s_n\right)}{\left(\Delta s_n - 4\Delta s_{n+1} + 6\Delta s_{n+2} - 4\Delta s_{n+3} + \Delta s_{n+4}\right)}. \tag{26}$$

The Thukral-sequence transformation Formula (6), for $k = 4$ is given as

$$t_4 = \frac{\sum_{i=0}^{4} (-1)^i \binom{4}{i} s_{n+i+1}\Delta s_{n+4-i}}{\sum_{i=0}^{4} (-1)^i \binom{4}{i} \Delta s_{n+4-i}}, \tag{27}$$

$$t_4 = \frac{\left(s_n\Delta s_{n+4} - 4s_{n+1}\Delta s_{n+3} + 6s_{n+2}\Delta s_{n+2} - 4s_{n+3}\Delta s_{n+1+}s_{n+4}\Delta s_n\right)}{\left(\Delta s_{n+4} - 4\Delta s_{n+3} + 6\Delta s_{n+2} - 4\Delta s_{n+1} + \Delta s_n\right)}. \tag{28}$$

It is apparent that (26) and (28) are identical and hence the Thukral-determinantal formula and the Thukral-sequence transformation formula produce identical results.

The Thukral-determinantal formula for $k = 5$:

$$T_5 = (-1)^5 \frac{\begin{vmatrix} \frac{1}{\Delta s_n} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+3}} & \frac{1}{\Delta s_{n+4}} & \frac{1}{\Delta s_{n+5}} \\ \frac{n}{\Delta s_n} & \frac{n+1}{\Delta s_{n+1}} & \frac{n+2}{\Delta s_{n+2}} & \frac{n+3}{\Delta s_{n+3}} & \frac{n+4}{\Delta s_{n+4}} & \frac{n+5}{\Delta s_{n+5}} \\ \frac{n^2}{\Delta s_n} & \frac{(n+1)^2}{\Delta s_{n+1}} & \frac{(n+2)^2}{\Delta s_{n+2}} & \frac{(n+3)^2}{\Delta s_{n+3}} & \frac{(n+4)^2}{\Delta s_{n+4}} & \frac{(n+5)^2}{\Delta s_{n+5}} \\ \frac{n^3}{\Delta s_n} & \frac{(n+1)^3}{\Delta s_{n+1}} & \frac{(n+2)^3}{\Delta s_{n+2}} & \frac{(n+3)^3}{\Delta s_{n+3}} & \frac{(n+4)^3}{\Delta s_{n+4}} & \frac{(n+5)^3}{\Delta s_{n+5}} \\ \frac{n^4}{\Delta s_n} & \frac{(n+1)^4}{\Delta s_{n+1}} & \frac{(n+2)^4}{\Delta s_{n+2}} & \frac{(n+3)^4}{\Delta s_{n+3}} & \frac{(n+4)^4}{\Delta s_{n+4}} & \frac{(n+5)^4}{\Delta s_{n+5}} \\ s_{n+5} & s_{n+4} & s_{n+3} & s_{n+2} & s_{n+1} & s_n \end{vmatrix}}{\begin{vmatrix} \frac{1}{\Delta s_{n+5}} & \frac{1}{\Delta s_{n+4}} & \frac{1}{\Delta s_{n+3}} & \frac{1}{\Delta s_{n+2}} & \frac{1}{\Delta s_{n+1}} & \frac{1}{\Delta s_n} \\ \frac{n}{\Delta s_{n+5}} & \frac{n+1}{\Delta s_{n+4}} & \frac{n+2}{\Delta s_{n+3}} & \frac{n+3}{\Delta s_{n+2}} & \frac{n+4}{\Delta s_{n+1}} & \frac{n+5}{\Delta s_n} \\ \frac{n^2}{\Delta s_{n+5}} & \frac{(n+1)^2}{\Delta s_{n+4}} & \frac{(n+2)^2}{\Delta s_{n+3}} & \frac{(n+3)^2}{\Delta s_{n+2}} & \frac{(n+4)^2}{\Delta s_{n+1}} & \frac{(n+5)^2}{\Delta s_n} \\ \frac{n^3}{\Delta s_{n+5}} & \frac{(n+1)^3}{\Delta s_{n+4}} & \frac{(n+2)^3}{\Delta s_{n+3}} & \frac{(n+3)^3}{\Delta s_{n+2}} & \frac{(n+4)^3}{\Delta s_{n+1}} & \frac{(n+5)^3}{\Delta s_n} \\ \frac{n^4}{\Delta s_{n+5}} & \frac{(n+1)^4}{\Delta s_{n+4}} & \frac{(n+2)^4}{\Delta s_{n+3}} & \frac{(n+3)^4}{\Delta s_{n+2}} & \frac{(n+4)^4}{\Delta s_{n+1}} & \frac{(n+5)^4}{\Delta s_n} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}}. \tag{29}$$

Similarly, we expand (29) and obtain

$$T_5 = -\frac{(s_n \Delta s_{n+5} - 5s_{n+1}\Delta s_{n+4} + 10s_{n+2}\Delta s_{n+3} - 10s_{n+3}\Delta s_{n+2} + 5s_{n+4}\Delta s_{n+1} - s_{n+5}\Delta s_n)\, R_5}{(\Delta s_n - 5\Delta s_{n+1} + 10\Delta s_{n+2} - 10\Delta s_{n+3} + 5\Delta s_{n+4} - \Delta s_{n+5})\, R_5}, \tag{30}$$

where the common factors of the numerator and the dominator of (30) are denoted as

$$R_5 = 288\,(\Delta s_{n+5}\Delta s_{n+4}\Delta s_{n+3}\Delta s_{n+2}\Delta s_{n+1}\Delta s_n)^{-1}. \tag{31}$$

By cancelling these factors and simplifying (31), we get

$$T_5 = \frac{(s_n \Delta s_{n+5} - 5s_{n+1}\Delta s_{n+4} + 10s_{n+2}\Delta s_{n+3} - 10s_{n+3}\Delta s_{n+2+}5s_{n+4}\Delta s_{n+1} - s_{n+5}\Delta s_n)}{(\Delta s_{n+5} - 5\Delta s_{n+4} + 10\Delta s_{n+3} - 10\Delta s_{n+2} + 5\Delta s_{n+1} - \Delta s_n)}. \tag{32}$$

Expanding the Thukral-sequence transformation Formula (6) for $k = 5$,

$$t_5 = \frac{\sum_{i=0}^{5} (-1)^i \binom{5}{i} s_{n+i+1}\Delta s_{n+k-i}}{\sum_{i=0}^{5} (-1)^i \binom{5}{i} \Delta s_{n+k-i}}, \tag{33}$$

$$t_5 = \frac{(s_n \Delta s_{n+5} - 5s_{n+1}\Delta s_{n+4} + 10s_{n+2}\Delta s_{n+3} - 10s_{n+3}\Delta s_{n+2} + 5s_{n+4}\Delta s_{n+1} - s_{n+5}\Delta s_n)}{(\Delta s_{n+5} - 5\Delta s_{n+4} + 10\Delta s_{n+3} - 10\Delta s_{n+2} + 5\Delta s_{n+1} - \Delta s_n)}. \tag{34}$$

We observe that the equations of the Thukral-determinantal formula and the Thukral-sequence transformation formula given by (32) and (34) respectively are identical.

In general, we define the Thukral-determinantal formula as

$$T_n^k = (-1)^k \frac{\begin{vmatrix} \dfrac{1}{\Delta s_n} & \dfrac{1}{\Delta s_{n+k-1}} & \cdots & \dfrac{1}{\Delta s_{n+k}} \\ \dfrac{n}{\Delta s_n} & \dfrac{n+1}{\Delta s_{n+k-1}} & \cdots & \dfrac{n+k}{\Delta s_{n+k}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+k} & s_{n+k-1} & \cdots & s_n \end{vmatrix}}{\begin{vmatrix} \dfrac{1}{\Delta s_{n+k}} & \dfrac{1}{\Delta s_{n+k-1}} & \cdots & \dfrac{1}{\Delta s_n} \\ \dfrac{n}{\Delta s_{n+k}} & \dfrac{n+k-1}{\Delta s_{n+k-1}} & \cdots & \dfrac{n+k}{\Delta s_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}}, \tag{35}$$

where $n,\ k \in \aleph$, provided the denominator of (35) is a nonzero and s_n is given by (2).

Mathematically we have demonstrated that the Thukral-determinantal Formula (35) produces identical results when compared to the Thukral-sequence transformation Formula (6), whereas only the odd values of k are identical to the formula of Brezinski-Durbin-Redivo-Zaglia sequence transformation, given by (5). Therefore, we postulate that the relationship between the two new Thukral's formulae is given as

$$T_n^k = t_n = \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} s_{n+i+1} \Delta s_{n+k-i}}{\sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta s_{n+k-i}}, \tag{36}$$

and we conjecture that the connection between the Thukral-determinantal formula and the Brezinski-Durbin-Redivo-Zaglia's sequence transformation is

$$T_n^{2k-1} = \frac{\sum_{i=0}^{2k-1} (-1)^i \binom{2k-1}{i} s_{n+i} \Delta s_{n+2k-i-1}}{\sum_{i=0}^{2k-1} (-1)^i \binom{2k-1}{i} \Delta s_{n+2k-i-1}}. \tag{37}$$

4. Remarks and Conclusion

In this study, we have introduced the Thukral-determinantal formula and the Thukral-sequence transformation formula for accelerating the convergence of a sequence. It is well established that the Durbin sequence transformation has not been presented in a ratio of two determinants [4]–[6]. Hence, it was of great interest to find a formula. In the process, we have demonstrated the similarities between the Thukral-determinantal formula for the Aitkin Δ^2 algorithm and the Durbin sequence transformation for case $k = 1$ and $k = 3$, respectively. Furthermore, we have found that the Thukral-determinantal formula produces mathematically equivalent results to the new Thukral-sequence transformation formula, and only odd values of k results are similar to Brezinski-Durbin-Redivo-Zaglia's sequence transformation. Finally, further work is in progress to obtain the variants of the Thukral-determinantal formula and establish an efficient convergence accelerator.

Conflict of Interest

Author declares that there is no conflict of interest.

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