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RESEARCH ARTICLE

Representations of Group Algebras of Non-Abelian Groups of Orders p^3, for a Prime $p \geq 3$

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ABSTRACT

In this paper, semidirect products are used to find the matrix representations of group algebras of non-abelian groups of order p^3, for a prime $p \geq 3$.

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1. Preliminaries

Let G be a group, assume that H is a normal subgroup of G, K is a subgroup of G, $H \cap K = \{1\}$, and $G = HK$. Suppose that K acts on H by automorphisms of H, then there exists a homomorphism $\varphi\colon K \to Aut(H)$. Assume the action is by conjugation, then for $k \in K$ and $h \in H$ we have $k.h = \varnothing(k)(h) = khk^{-1}$. G is an internal semidirect product of H and K by φ, it is denoted by $G = H \rtimes_{\varphi} K$ [1].

Non-abelian groups of orders p^3, for a prime $p \geq 3$ are of two types [1]:

$$G_1 = C_{p^2}\langle\alpha\rangle \rtimes_{\varphi} C_p\langle\beta\rangle \quad \text{and} \quad G_2 = (C_p\langle\alpha\rangle \times C_p\langle\beta\rangle) \rtimes_{\varphi} C_p\langle\gamma\rangle.$$

Thus,

$$G_1 = \left\langle \alpha, \beta\colon \alpha^{p^2} = \beta^p = 1, \beta\alpha = \alpha^{1+p}\beta \right\rangle$$

and

$$G_2 = \left\langle \alpha, \beta, \gamma\colon \alpha^p = \beta^p = \gamma^p = 1, \alpha\beta = \beta\alpha, \gamma\beta = \beta\gamma, \gamma\alpha = \alpha\beta\gamma \right\rangle$$

Let F be a field. A ring A with unity is an algebra over F (breifly F-algebra) if A is a vector space over F and the following compatibility condition holds $(sa).b = s(a.b) = a.(sb)$ for any $a, b \in A$ and any $s \in F$. A is also called associative algebra (over F). The dimension of the algebra A is the dimension of A as a vector space over F.

Theorem 1 [2]
Let A be a n-dimensional algebra over a field F. Then there is a one-to-one algebra homomorphism from A into $M_n(F)$, the algebra of n-matrices over F.

Let $G = \{g_1 = 1, g_2, \ldots, g_n\}$ be a finite group of order n and F a field. Define $FG = \{a_1g_1 + a_2g_2 + \ldots + a_ng_n\colon a_i \in F\}$. FG is n-dimensional vector space over F with basis G. Multiplication of G can be extended linearly to FG. Thus, FG becomes an algebra over F of dimension n. FG is called group algebra. The following identifications should be realized:

(i) $0_F g_G = 0_{FG} = 0$ for any $g \in G$.
(ii) $1_F g_G = g_{FG} = g$ for any $g \in G$. In particular $1_F 1_G = 1_{FG} = 1$.
(iii) $a_F 1_G = a_{FG}$ for any $a \in F$.

A circulant matrix M on parameters $a_0, a_1, \ldots, a_{n-1}$ is defined as follows:

$$M(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_{n-1} \cdots & a_1 \\ a_1 & a_0 \ \cdots & a_2 \\ \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} \cdots & a_0 \end{bmatrix}$$

This matrix may be denoted in terms of its columns by $[col(a_0)|col(a_{n-1})|\ldots|col(a_1)]$.

M is said to be circulant block matrix if it is of the form $M(M_1, M_2, \ldots, M_n)$. i.e., it is circulant blockwise on the blocks $M_1, M_2, \ldots, M_n$.

Thus,

$$M = \begin{bmatrix} M_1 & M_n \cdots & M_2 \\ M_2 & M_1 \cdots & M_3 \\ \vdots & \vdots & \vdots \\ M_n & M_{n-1} \cdots & M_1 \end{bmatrix}.$$

2. Main Results

Theorem 2 [3]

Let F be a field and $G = \langle \alpha : \alpha^n = 1 \rangle$ a cyclic group of order n. Then any element $a_0 1 + a_1\alpha + \ldots + a_{n-1}\alpha^{n-1}$ of FG can be represented with respect to the ordered basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ by the circulant matrix $M(a_0, a_1, \ldots, a_{n-1})$.

Proof

Let $w = a_0 1 + a_1\alpha + \ldots + a_{n-1}\alpha^{n-1}$ be in FG. $w\alpha = a_0\alpha + a_1\alpha^2 + \ldots + a_{n-1}1 = a_{n-1}1 + a_0\alpha + \ldots + a_{n-2}\alpha^{n-1} \ldots w\alpha^{n-1} = a_0\alpha^{n-1} + a_1 1 + \ldots + a_{n-1}\alpha^{n-2} = a_1 1 + a_2\alpha + \ldots + a_0\alpha^{n-1}$. Then the matrix representation of w with respect to the basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is $\begin{bmatrix} a_0 & a_{n-1} \cdots & a_1 \\ a_1 & a_0 \ \cdots & a_2 \\ \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} \cdots & a_0 \end{bmatrix}$ which is $M(a_0, a_1, \ldots, a_{n-1})$.

Note that if the order of the basis elements is changed, we obtain a different matrix of representation. The new matrix is obtained by suitable interchanging of the columns of the matrix $M(a_0, a_1, \ldots, a_{n-1})$. In [4] the representation is done for the non-split metacyclic group.

For more complicated finite groups we use the circulant block matrices to do the required representations.

Now, let G be an internal semidirect product of H and a cyclic group $K = \langle \gamma \rangle$ by φ.

Then the matrix representation $[w]$ of the general element w in FG is given as follows:

$G = H \rtimes_\varphi K$, $\varphi : K \to Aut(H)$ is a homomorphism, $\varphi(\gamma)(h) = \gamma h \gamma^{-1}$. Suppose that $H = \{h_1, h_2, \ldots, h_n\}$, $K = C_m\langle \gamma \rangle = \{1, \gamma, \ldots, \gamma^{m-1}\}$ then the general element w in FG is $w = a_1h_1 1 + a_2h_2 1 + \ldots + a_nh_n 1 + a_{n+1}h_1\gamma + a_{n+2}h_2\gamma + \ldots + a_{2n}h_n\gamma + a_{2n+1}h_1\gamma^2 + \ldots + a_{3n}h_n\gamma^2 + \ldots + a_{mn}h_n\gamma^{m-1}$. Now we can write w as:

$$w = w_1 + w_2 + \ldots + w_m,$$

where

$$w_1 = a_1h_1 1 + a_2h_2 1 + \ldots + a_nh_n 1$$

$$w_2 = a_{n+1}h_1\gamma + a_{n+2}h_2\gamma + \ldots + a_{2n}h_n\gamma$$

$$\vdots$$

$$w_m = a_{(m-1)(n+1)}h_1\gamma^{m-1} + \ldots + a_{mn}h_n\gamma^{m-1}$$

The matrix representation $[w]$ of w is $[w] = M\left([w_1], [w_2]^\gamma, \ldots, [w_m]^{\gamma^{m-1}}\right)$, where $\gamma^i : H \to H$ is the automorphism $\gamma^i = \varphi(\gamma)(h) = \gamma^i h \gamma^{-i}$ and $[w_i] = [col(h_1)|col(h_2)|\ldots|col(h_n)]$, $[w_i]^{\gamma^i} = \left[col\left(\gamma^i(h_1)\right)\left|col\left(\gamma^i(h_2)\right)\right|\ldots\left|col\left(\gamma^i(h_n)\right)\right|\right]$. Thus, we get the following theorem:

Theorem 3

With the above notations, the matrix representation $[w]$ of the general element w in FG.

$$[w]=\begin{bmatrix} [w_1] & [w_m]^{\gamma^{m-1}} & \dots & [w_2]^{\gamma} \\ [w_2]^{\gamma} & [w_1] & \dots & [w_m]^{\gamma^2} \\ \vdots & \vdots & \vdots & \vdots \\ [w_m]^{\gamma^{m-1}} & [w_m]^{\gamma^{m-2}} & & [w_1] \end{bmatrix}.$$

3. Applications

Finally, we use theorem 3 to compute the matrix representations of FG_1 and FG_2, when the prime $p=3$.

1) $G_1=\left\langle \alpha,\beta : \alpha^{3^2}=\beta^3=1, \beta\alpha=\alpha^{1+3}\beta\right\rangle$

$=\left\{1,\alpha,\alpha^2,\dots,\alpha^8,\beta,\alpha\beta,\alpha^2\beta,\dots,\alpha^8\beta,\beta^2,\alpha\beta^2,\alpha^2\beta^2,\dots,\alpha^8\beta^2\right\}$.

The general element of FG_1 is $w=a_0 1+a_1\alpha+\dots+a_8\alpha^8+a_9\beta+a_{10}\alpha\beta+\dots+a_{17}\alpha^8\beta+a_{18}\beta^2+a_{19}\alpha\beta^2+\dots+a_{26}\alpha^8\beta^2$. Let $w_1=a_0 1+a_1\alpha+\dots+a_8\alpha^8$, $w_2=a_9\beta+a_{10}\alpha\beta+\dots+a_{17}\alpha^8\beta$, $w_3=a_{18}\beta^2+a_{19}\alpha\beta^2+\dots+a_{26}\alpha^8\beta^2$. Then $w=w_1+w_2+w_3$.

By theorem 3, matrix representation of w is $[w]=\begin{bmatrix} [w_1] & [w_3]^{\beta^2} & [w_2]^{\beta} \\ [w_2]^{\beta} & [w_1] & [w_3]^{\beta^2} \\ [w_3]^{\beta^2} & [w_2]^{\beta} & [w_1] \end{bmatrix}$

$$[w_1]=\begin{bmatrix} a_0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_8 & a_7 & a_6 & a_5 & a_4 \\ a_4 & a_3 & a_2 & a_1 & a_0 & a_8 & a_7 & a_6 & a_5 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & a_8 & a_7 & a_6 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & a_8 & a_7 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & a_8 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

$G_1 = C_9\langle\alpha\rangle \rtimes_{\varphi} C_3\langle\beta\rangle$, $\varphi : C_3\langle\beta\rangle \to Aut(C_9\langle\alpha\rangle)$ is a homomorphism such that $\varphi(\beta)(\alpha)=\beta\alpha^3\beta$

$\varphi(\beta)(1)=\beta 1\beta^{-1}=1$, $\varphi(\beta)(\alpha)=\beta\alpha\beta^{-1}=\alpha^4$, $\varphi(\beta)\left(\alpha^2\right)=\beta\alpha^2\beta^{-1}=\alpha^8$, $\varphi(\beta)\left(\alpha^3\right)=\beta\alpha^3\beta=\alpha^3$, $\varphi(\beta)\left(\alpha^4\right)=\beta\alpha^4\beta^{-1}=\alpha^7$, $\varphi(\beta)\left(\alpha^5\right)=\beta\alpha^5\beta^{-1}=\alpha^2$, $\varphi(\beta)\left(\alpha^6\right)=\beta\alpha^6\beta^{-1}=\alpha^6$, $\varphi(\beta)\left(\alpha^7\right)=\beta\alpha^7\beta^{-1}=\alpha$, $\varphi(\beta)\left(\alpha^8\right)=\beta\alpha^8\beta^{-1}=\alpha^5$.

$$[w_2]=\left[col(1)\,|col(\alpha)|\,col\left(\alpha^2\right)\left|col\left(\alpha^3\right)\right|col\left(\alpha^4\right)\left|col\left(\alpha^5\right)\right|col\left(\alpha^6\right)|col\left(\alpha^7\right)|col\left(\alpha^8\right)\right]$$

$$[w_2]^{\beta}=\left[col(1)\left|col\left(\alpha^4\right)\right|col\left(\alpha^8\right)\left|col\left(\alpha^3\right)\right|col\left(\alpha^7\right)\left|col\left(\alpha^2\right)\right|col\left(\alpha^6\right)|col(\alpha)\,|col\left(\alpha^5\right)\right]$$

$$[w_2]^{\beta}=\left[\begin{array}{ccc|ccc|ccc} a_9 & a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} & a_{17} & a_{13} \\ a_{10} & a_{15} & a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 & a_{14} \\ a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} \\ \hdashline a_{12} & a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} & a_{11} & a_{16} \\ a_{13} & a_9 & a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} & a_{17} \\ a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 \\ \hdashline a_{15} & a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 & a_{14} & a_{10} \\ a_{16} & a_{12} & a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} & a_{11} \\ a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} \end{array}\right]$$

$$\varphi\left(\beta^2\right)(\alpha)=\beta^2\alpha\beta^{-2}$$

$\varphi\left(\beta^2\right)(1)=\beta^2 1\beta^{-2}=1$, $\varphi\left(\beta^2\right)(\alpha)=\beta^2\alpha\beta^{-2}=\alpha^7$, $\varphi\left(\beta^2\right)\left(\alpha^2\right)=\beta^2\alpha^2\beta^{-2}=\alpha^5$, $\varphi\left(\beta^2\right)\left(\alpha^3\right)=\beta^2\alpha^3\beta^{-2}=\alpha^3$, $\varphi\left(\beta^2\right)\left(\alpha^4\right)=\beta^2\alpha^4\beta^{-2}=\alpha, \varphi\left(\beta^2\right)\left(\alpha^5\right)=\beta^2\alpha^5\beta^{-2}=\alpha^8$, $\varphi\left(\beta^6\right)\left(\alpha^6\right)=\beta^2\alpha^6\beta^{-2}=\alpha^6$, $\varphi\left(\beta^2\right)\left(\alpha^7\right)=\beta^2\alpha^7\beta^{-2}=\alpha^4$, $\varphi\left(\beta^2\right)\left(\alpha^8\right)=\beta^2\alpha^8\beta^{-2}=\alpha^2$.

$$[w_3]=\left[col(1)\,|col(\alpha)|\,col\left(\alpha^2\right)\left|col\left(\alpha^3\right)\right|col\left(\alpha^4\right)\left|col\left(\alpha^5\right)\right|col\left(\alpha^6\right)|col\left(\alpha^7\right)|col\left(\alpha^8\right)\right]$$

$$[w_3]^{\beta^2}=\left[col(1)\left|col\left(\alpha^7\right)\right|col\left(\alpha^5\right)\left|col\left(\alpha^3\right)\right|col(\alpha)\left|col\left(\alpha^8\right)\right|col\left(\alpha^6\right)|col\left(\alpha^4\right)|col\left(\alpha^2\right)\right]$$

$$[w_3]^{\beta^2} = \left[\begin{array}{ccc|ccc|ccc} a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} & a_{23} & a_{25} \\ a_{19} & a_{21} & a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} \\ a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} \\ \hline a_{21} & a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{19} \\ a_{22} & a_{24} & a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} & a_{20} \\ a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} \\ \hline a_{24} & a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} & a_{20} & a_{22} \\ a_{25} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} & a_{23} \\ a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} \end{array}\right]$$

2) $G_2 = \langle \alpha, \beta, \gamma : \alpha^3 = \beta^3 = \gamma^3 = 1, \alpha\beta = \beta\alpha, \gamma\beta = \beta\gamma, \gamma\alpha = \alpha\beta\gamma \rangle$ $G_2 = \{1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta, \beta^2, \alpha\beta^2, \alpha^2\beta^2, \gamma, \alpha\gamma, \alpha^2\gamma, \beta\gamma, \alpha\beta\gamma, \alpha^2\beta\gamma, \beta^2\gamma, \alpha\beta^2\gamma, \alpha^2\beta^2\gamma, \gamma^2, \alpha\gamma^2, \alpha^2\gamma^2, \beta\gamma^2, \alpha\beta\gamma^2, \alpha^2\beta\gamma^2, \beta^2\gamma^2, \alpha\beta^2\gamma^2, \alpha^2\beta^2\gamma^2\}$

The general element of FG_2 is $w = a_0 1 + a_1\alpha + a_2\alpha^2 + a_3\beta + a_4\alpha\beta + a_5\alpha^2\beta + a_6\beta^2 + a_7\alpha\beta^2 + a_8\alpha^2\beta^2 + a_9\gamma + a_{10}\alpha\gamma + a_{11}\alpha^2\gamma + a_{12}\beta\gamma + a_{13}\alpha\beta\gamma + a_{14}\alpha^2\beta\gamma + a_{15}\beta^2\gamma + a_{16}\alpha\beta^2\gamma + a_{17}\alpha^2\beta^2\gamma + a_{18}\gamma^2 + a_{19}\alpha\gamma^2 + a_{20}\alpha^2\gamma^2 + a_{21}\beta\gamma^2 + a_{22}\alpha\beta\gamma^2 + a_{23}\alpha^2\beta\gamma^2 + a_{24}\beta^2\gamma^2 + a_{25}\alpha\beta^2\gamma^2 + a_{26}\alpha^2\beta^2\gamma^2$.

$$w_1 = a_0 1 + a_1\alpha + a_2\alpha^2 + a_3\beta + a_4\alpha\beta + a_5\alpha^2\beta + a_6\beta^2 + a_7\alpha\beta^2 + a_8\alpha^2\beta^2$$

$$w_2 = a_9\gamma + a_{10}\alpha\gamma + a_{11}\alpha^2\gamma + a_{12}\beta\gamma + a_{13}\alpha\beta\gamma + a_{14}\alpha^2\beta\gamma + a_{15}\beta^2\gamma + a_{16}\alpha\beta^2\gamma$$

$$w_3 = a_{18}\gamma^2 + a_{19}\alpha\gamma^2 + a_{20}\alpha^2\gamma^2 + a_{21}\beta\gamma^2 + a_{22}\alpha\beta\gamma^2 + a_{23}\alpha^2\beta\gamma^2 + a_{24}\beta^2\gamma^2 + a_{25}\alpha\beta^2\gamma^2 + a_{26}\alpha^2\beta^2\gamma^2.$$

Then $w = w_1 + w_2 + w_3$.

The matrix representation of w is $[w] = \begin{bmatrix} [w_1] & [w_3]^{\gamma^2} & [w_2]^{\gamma} \\ [w_2]^{\gamma} & [w_1] & [w_3]^{\gamma^2} \\ [w_3]^{\gamma^2} & [w_2]^{\gamma} & [w_1] \end{bmatrix}$

$$[w_1] = \left[\begin{array}{ccc|ccc|ccc} a_0 & a_2 & a_1 & a_6 & a_8 & a_7 & a_3 & a_5 & a_4 \\ a_1 & a_0 & a_2 & a_7 & a_6 & a_8 & a_4 & a_3 & a_5 \\ a_2 & a_1 & a_0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ \hline a_3 & a_5 & a_4 & a_0 & a_2 & a_1 & a_6 & a_8 & a_7 \\ a_4 & a_3 & a_5 & a_1 & a_0 & a_2 & a_7 & a_6 & a_8 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & a_8 & a_7 & a_6 \\ \hline a_6 & a_8 & a_7 & a_3 & a_5 & a_4 & a_0 & a_2 & a_1 \\ a_7 & a_6 & a_8 & a_4 & a_3 & a_5 & a_1 & a_0 & a_2 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{array}\right]$$

$G_2 = (C_3\langle\alpha\rangle \times C_3\langle\beta\rangle) \rtimes_{\varphi} C_3\langle\gamma\rangle$, $\varphi : C_3\langle\gamma\rangle \to Aut(C_3\langle\alpha\rangle \times C_3\langle\beta\rangle)$ is a homomorphism such that $\varphi(\gamma)(\alpha) = \gamma\alpha\gamma^{-1}$.

$\varphi(\gamma)(1) = \gamma 1\gamma^{-1} = 1$, $\varphi(\gamma)(\alpha) = \gamma\alpha\gamma^{-1} = \alpha\beta$, $\varphi(\gamma)(\alpha^2) = \gamma\alpha^2\gamma^{-1} = \alpha^2\beta^2$, $\varphi(\gamma)(\beta) = \gamma\beta\gamma^{-1} = \beta$, $\varphi(\gamma)(\alpha\beta) = \gamma\alpha\beta\gamma^{-1} = \alpha\beta^2$, $\varphi(\gamma)(\alpha^2\beta) = \gamma\alpha^2\beta\gamma^{-1} = \alpha^2$, $\varphi(\gamma)(\beta^2) = \gamma\beta^2\gamma^{-1} = \beta^2$, $\varphi(\gamma)(\alpha\beta^2) = \gamma\alpha\beta^2\gamma^{-1} = \alpha$, $\varphi(\gamma)(\alpha^2\beta^2) = \gamma\alpha^2\beta^2\gamma^{-1} = \alpha^2\beta$.

$$[w_2] = \left[col(1)\,|col(\alpha)|\,col(\alpha^2)\,|col(\beta)|\,col(\alpha\beta)\,|col(\alpha^2\beta)|\,col(\beta^2)\,|col(\alpha\beta^2)\,|col(\alpha^2\beta^2)\right]$$

$$[w_2]^{\gamma} = \left[col(1)\,|col(\alpha\beta)|\,col(\alpha^2\beta^2)\,|col(\beta)|\,col(\alpha\beta^2)\,|col(\alpha^2)|\,col(\beta^2)\,|col(\alpha)\,|col(\alpha^2\beta)\right]$$

$$[w_2]^{\gamma} = \left[\begin{array}{ccc|ccc|ccc} a_9 & a_{17} & a_{13} & a_{15} & a_{14} & a_{10} & a_{12} & a_{11} & a_{16} \\ a_{10} & a_{15} & a_{14} & a_{16} & a_{12} & a_{11} & a_{13} & a_9 & a_{17} \\ a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} \\ \hline a_{12} & a_{11} & a_{16} & a_9 & a_{17} & a_{13} & a_{15} & a_{14} & a_{10} \\ a_{13} & a_9 & a_{17} & a_{10} & a_{15} & a_{14} & a_{16} & a_{12} & a_{11} \\ a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} & a_{17} & a_{13} & a_9 \\ \hline a_{15} & a_{14} & a_{10} & a_{12} & a_{11} & a_{16} & a_9 & a_{17} & a_{13} \\ a_{16} & a_{12} & a_{11} & a_{13} & a_9 & a_{17} & a_{10} & a_{15} & a_{14} \\ a_{17} & a_{13} & a_9 & a_{14} & a_{10} & a_{15} & a_{11} & a_{16} & a_{12} \end{array}\right]$$

$$\varphi\left(\gamma^2\right)(\alpha) = \gamma^2\alpha\gamma^{-2}$$

$\varphi\left(\gamma^2\right)(1) = \gamma^2 1\gamma^{-2} = 1$, $\varphi\left(\gamma^2\right)(\alpha) = \gamma^2\alpha\gamma^{-2} = \alpha\beta^2$, $\varphi\left(\gamma^2\right)\left(\alpha^2\right) = \gamma^2\alpha^2\gamma^{-2} = \alpha^2\beta$, $\varphi\left(\gamma^2\right)(\beta) = \gamma^2\beta\gamma^{-2} = \beta$, $\varphi\left(\gamma^2\right)(\alpha\beta) = \gamma^2\alpha\beta\gamma^{-2} = \alpha$, $\varphi\left(\gamma^2\right)\left(\alpha^2\beta\right) = \gamma^2\alpha^2\beta\gamma^{-2} = \alpha^2\beta^2$, $\varphi\left(\gamma^2\right)\left(\beta^2\right) = \gamma^2\beta^2\gamma^{-2} = \beta^2$, $\varphi\left(\gamma^2\right)\left(\alpha\beta^2\right) = \gamma^2\alpha\beta^2\gamma^{-2} = \alpha\beta$, $\varphi\left(\gamma^2\right)\left(\alpha^2\beta^2\right) = \gamma^2\alpha^2\beta^2\gamma^{-2} = \alpha^2$.

$$[w_3] = \left[col(1)\,|col(\alpha)|\,col\left(\alpha^2\right)|col(\beta)|\,col(\alpha\beta)\,|col\left(\alpha^2\beta\right)|\,col\left(\beta^2\right)|col\left(\alpha\beta^2\right)|col(\alpha^2\beta^2)\right]$$

$$[w_3]^{\gamma^2} = \left[col(1)\,|col\left(\alpha\beta^2\right)|\,col\left(\alpha^2\beta\right)|col(\beta)|\,col(\alpha)\,|col\left(\alpha^2\beta^2\right)|\,col\left(\beta^2\right)|col(\alpha\beta)\,|col(\alpha^2)\right]$$

$$[w_3]^{\gamma^2} = \left[\begin{array}{ccc|ccc|ccc} a_{18} & a_{23} & a_{25} & a_{24} & a_{20} & a_{22} & a_{21} & a_{26} & a_{19} \\ a_{19} & a_{21} & a_{26} & a_{25} & a_{18} & a_{23} & a_{22} & a_{24} & a_{20} \\ a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} \\ \hline a_{21} & a_{26} & a_{19} & a_{18} & a_{23} & a_{25} & a_{24} & a_{20} & a_{22} \\ a_{22} & a_{24} & a_{20} & a_{19} & a_{21} & a_{26} & a_{25} & a_{18} & a_{23} \\ a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{19} & a_{21} \\ \hline a_{24} & a_{20} & a_{22} & a_{21} & a_{26} & a_{19} & a_{18} & a_{23} & a_{25} \\ a_{25} & a_{18} & a_{23} & a_{22} & a_{24} & a_{20} & a_{19} & a_{21} & a_{26} \\ a_{26} & a_{19} & a_{21} & a_{23} & a_{25} & a_{18} & a_{20} & a_{22} & a_{24} \end{array}\right]$$

For greater prime p, the same method may be applied.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

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