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RESEARCH ARTICLE

Iterative Procedure for Finite Family of Total Asymptotically Nonexpansive Maps (TAN)

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ABSTRACT

In this paper, CQ Algorithms for iterative approximation of a common fixed point of a finite family of nonlinear maps were introduced and sufficient conditions for the strong convergence of this process to a common fixed point of the family of Total asymptotically Nonexpansive maps (TAN) were proved.

Keywords: CQ algorithm, strong convergence, Total Asymptotically Nonexpansive Maps (TAN).

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1. Preliminaries

Let H be a normed space, K be a nonempty closed convex subset of H and $T : K \to K$ be a map. The mapping T is called *asymptotically nonexpansive mapping* if and only if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x.y \in K$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \quad \forall\, n \in \mathbb{N} \tag{1}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalisation of nonexpansive mappings. As further generalisation of class of nonexpansive mappings, Alber *et al*. [2] introduced the class of total asymptotically nonexpansive mappings, where a mapping $T : K \longrightarrow K$ is called *total asymptotically nonexpansive* (TAN) if and only if there exist two sequences $\{\mu_n\}_{n\geq1}, \{\eta_n\}_{n\geq1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continuous function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n\phi(\|x - y\|) + \eta_n \quad n \geq 1 \tag{2}$$

In Ofoedu and Nnubia [3], an example to show that the class of asymptotically nonexpansive mappings is properly contained in the class of total asymptotically nonexpansive mappings was given. The class of asymptotically nonexpansive type mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense. These classes of mappings had been studied extensively by several authors (see e.g., [4]–[9]).

A map T is said to satisfies *condition B* if there exists $f : [0, \infty) \to [0, \infty)$ strictly increasing, continuous, $f(0) = 0, f(r) > 0\ \forall\ r > 0$ such that for all $x \in D(T), \|x - Tx\| \geq f(d(x, F))$ where $F = F(T) = \{x \in D(T) : x = Tx\}$ and $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Lemma 1.1 *Takahashi [10]*

Let $\{\mu_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditons: $\sum_{n\geq 0}^{\infty} \beta_n = \infty$, $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\gamma_n = 0(\beta_n)$. Suppose that

$$\mu_{n+1}^2 \leq \mu_n^2 - \beta_n\psi(\mu_{n+1}) + \gamma_n; \quad n = 1, 2, ...$$

where $\psi : [0,1) \to [0,1)$ is a strictly increasing function with $\psi(0) = 0$. Then $\mu_n \to 0$ as $n \to \infty$.

Lemma 1.2 *Moore and Nnoli [11]*

Let K be closed convex nonempty subset of a real Hilbert Space H. Let $\{x_n\}$ be a sequence in H, $y \in H$ and $z = P_K y$ be such that $\omega_w(x_n) \subseteq K$ and $\|x_n - y\| \le \|y - z\|$ $\forall n \ge 1$, then $\{x_n\}$ converges strongly to z.

$$\omega_w(x_n) = \{z \in H : \exists \{x_{nj(z)}\} \subset \{x_n\} \ni x_{nj(z)} \to^w z \text{ as } j \to \infty\}$$

Lemma 1.3 *Ofoedu and Nnubia [9, p. 703]*

Let E be a reflexive Banach space with weakly continuous normalized duality mapping. Let K be a closed convex subset of E and $T : K \to K$ a uniformly continuous total asymptotically nonexpansive mapping with bounded orbits. Then $I - T$ is demiclosed at zero.

Proposition 1.1 *Ofoedu and Nnubia [9, p. 704]*

Let H be a real Hilbert space, let K be a nonempty closed convex subset of H and let $T_i : K \longrightarrow K$, where $i \in I = \{1, 2, ..., m\}$, be m uniformly continuous total asymptotically nonexpansive mappings from K into itself with sequences $\{\mu_{n,i}\}_{n\ge1}, \{\eta_{n,i}\}_{n\ge1} \subset [0, +\infty)$ such that $\lim_{n\to\infty} \mu_{n,i} = 0 = \lim_{n\to\infty} \eta_{n,i}$ and with function $\phi_i : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi_i(t) \le M_0 t \quad \forall \quad t > M_1$ for some constants $M_0, M_1 > 0$. Let $\mu_n = \max_{i\in I}\{\mu_{n,i}\}$ and $\eta_n = \max_{i\in I}\{\eta_{n,i}\}$ and, $\phi(t) = \max_{i\in I}\{\phi_i(t)\} \forall\ t \in [0, \infty)$. Suppose that $F(T) = \bigcap_{i=1}^m F(T_i)$, then $F(T)$ is closed and convex.

Proposition 1.2 *Nnubia and Bishop [6, p. 74]*

Let K be a nonempty subset of a real normed space E and $T_i : K \longrightarrow K$, where $i \in I = \{1, 2, ..., m\}$, be m total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_n\}_{n\ge1}, \{\eta_n\}_{n\ge1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continuous function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n; \quad n \ge 1, \forall\ i \in I. \tag{3}$$

2. Main Result

Proposition 2.1 *Result*

Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \le ct \forall\ t \ge k$, then T is total asymptotically nonexpansive if $\exists \nu_n = \mu_n c$ and $\gamma_n = \mu_n c_0 + \eta_n$ such that

$$\|T^n x - T^n y\| \le (1 + \nu_n)\|x - y\| + \gamma_n$$

Proof

Suppose T is total asymptotically nonexpansive, that is, let T be such that

$$\|T^n x - T^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n \quad n \ge 1 \tag{4}$$

Since ϕ is continuous, it follows that ϕ attains its maximum (say c_0) on the interval $[0, k]$; moreover, $\phi(t) \le ct$ whenever $t > k$. Thus,

$$\phi(t) \le c_0 + ct \ \ \forall\ t \in [0, +\infty). \tag{5}$$

So, we have,

$$\begin{aligned} \|T^n x - T^n y\| &\le \|x - y\| + \mu_n(c_0 + c\|x - y\|) + \eta_n \quad n \ge 1 \\ &= (1 + \mu_n c)\|x - y\| + \mu_n c_0 + \eta_n \\ &= (1 + \nu_n)\|x - y\| + \gamma_n \end{aligned}$$

where $\nu_n = \mu_n c$ and $\gamma_n = \mu_n c_0 + \eta_n$ Thus completing the proof.

Theorem 2.1 Let H, K, T_i, and F be as in Propostion 1.1, then $\{x_n\}_{n\ge1}$ generated iteratively by:

$$\begin{aligned} y_n &= (1 - \alpha_n)x_n + \alpha_n T_{i(n)}^{m(n)} x_n \ \ ; i(n) \equiv n \mod m \ \forall\ n \in \mathbb{Z}; m(n) = 1 + \left[\frac{n}{m}\right] \\ K_n &= \{z \in K : \ \|y_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_{i(n)}^{m(n)} x_n\|^2 + \sigma_n\} \\ Q_n &= \{z \in K : \ \langle x_n - z, x_o - x_n\rangle \ge 0\} \\ x_{n+1} &= P_{K_n \cap Q_n} x_0. \end{aligned} \tag{6}$$

converges to $P_F x_0$ where $\sigma_n = \alpha_n(k_{m(n),i(n)}^2 - 1)(diam.K)^2 + \alpha_n(2k_{m(n),i(n)} diam.K + \nu_{m(n),i(n)})\nu_{m(n),(n)}$ and $\{\alpha_n\} \subset [a, b] \subset (0, 1)$.

Proof

Let $x^* \in F$.

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\alpha_n)(x_n - x*) + \alpha_n(T_{i(n)}^{m(n)}x_n - x^*)\|^2 \\
&\leq (1-\alpha_n)\|x_n - x*\|^2 + \alpha_n\|T_{i(n)}^{m(n)}x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - x*\|^2 + \alpha_n(1+k_{i(n)m(n)})^2\|x_n - x^*\|^2 + (2(1+k_{i(n)m(n)})\|x_n - x^*\| \\
&\quad + v_{i(n)m(n)})v_{i(n)m(n)} - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \\
&= (1+\alpha_n[1+k_{i(n)m(n)} - 1])\|x_n - x^*\|^2 + (2(1+k_{i(n)m(n)})\|x_n - x^*\| \\
&\quad + v_{i(n)m(n)})v_{i(n)m(n)} - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \\
&\leq \|x_n - x*\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \\
&\quad + \alpha_n[(1+k_{i(n)m(n)})^2 - 1](diam.K)^2 + \alpha_n[2(1+k_{i(n)m(n)})(diam.K) \\
&\quad + v_{i(n)m(n)}]v_{i(n)m(n)} \\
&= \|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 + \sigma_n.
\end{aligned}$$

So that $x^* \in K_n \quad \forall n$. Hence, $F \subset K_n \; \forall n$. For $n = 0, \quad Q_o = K$. $F \subset Q_o$.

Let $F \subset Q_v$, we show that $F \subset Q_{v+1}$. Now, x_{v+1} is the projection of x_o onto $K_v \cap Q_v$. then (i) $\langle x_{v+1} - z\,, x_o - x_{v+1}\rangle \geq 0.\forall z \in K_v \cap Q_v$.

Since, $F \subset K_v \cap Q_v$, then $\langle x_{v+1} - x^*\,, x_o - x_{v+1}\rangle \geq 0 \; \forall x^* \in F$. So, $F \subset Q_{v+1}$ and hence $F \subset Q_n$ $\forall n \geq 0$.

Now,

$$\|x_o - P_{Q_n}x_o\| \leq \|x_o - y\| \;\; \forall \; y \in Q_n$$

Hence, $\forall n$

$$\|x_n - x_o\| = \|x_o - P_{Q_n}x_o\| \leq \|x_o - y\| \;\; \forall \; y \in Q_n.$$

and since $F \subset Q_n$, then $\|x_n - x_o\| \leq \|x_o - x^*\| \;\; \forall \; x^* \in F$.

In particular, $\|x_n - x_o\| \leq \|x_o - x^*\|; \quad x^* = P_F x_o$.

Since, $x_{n+1} \in Q_n$, $\langle x_{n+1} - x_n\,,\; x_n - x_o\rangle \geq 0$.

So,

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_o - (x_n - x_o)\|^2 \\
&= \|x_{n+1} - x_o^*\|^2 - \|x_n - x_o\|^2 - 2\,\langle x_{n+1} - x_n\,,\; x_n - x_o\rangle \\
&\leq \|x_{n+1} - x_o\|^2 - \|x_n - x_o\|^2
\end{aligned}$$

Then, $\|x_n - x_o\| \leq \|x_{n+1} - x_o\|$.

Since $\{\|x_n - x_o\|\}$ is bounded, then $\lim\limits_{n\to\infty} \|x_n - x_o\|$ exists.

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{7}$$

Observe that by (7)

$$\lim_{n\to\infty} \|x_{n+i} - x_n\| = 0 = \lim_{n\to\infty} \|x_{n-i} - x_n\| \forall \; i \in \{1, ..., m\}. \tag{8}$$

Now,

$$\begin{aligned}
\alpha_n^2\|x_n - T_{i(n)}^{m(n)}x_n\|^2 &= \|y_n - x_n\|^2 \\
&\leq (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n)^2 \\
&= \|y_n - x_{n+1}\|^2 + 2\|y_n - x_{n+1}\|.\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2
\end{aligned}$$

$x_{n+1} \in K_n$, so that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 + \sigma_n.$$

Hence,

$$\alpha_n^2\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_{i(n)}^{m(n)}x_n\|^2 + \sigma_n$$
$$+ 2\|y_n - x_{n+1}\|.\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2$$

and hence, $\alpha_n^2\|x_n - T_{i(n)}^{m(n)}x_n\|^2 \leq 2\|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|.\|x_{n+1} - x_n\| + \sigma_n$
Thus,

$$\lim_{n\to\infty} \|x_n - T_{i(n)}^{m(n)}x_n\| = 0 \tag{9}$$

Now, $\forall\ n > m$, we have $n = (n-m)(modm)$ and since $n = (m(n)-1)m + i(n)$, we obtain $n - m = (m(n)-1)N + i(n) - m = (m(n-m)-1) + i(n-m)$, so that $n - m = [(m(n)-1)-1]m + i(n) = (m(n-m)-1)m + i(n-m)$. Hence, $m(n) - 1 = m(n-m)$ and $i(n) = i(n-m)$. Similarly, $m(n+1-m) = m(n+1) - 1$ and i(n + 1 − N) = i(n + 1). Using this we obtain

$$\begin{aligned} \|x_n - T_{i(n+1)}x_{n+1}\| &\leq \|x_n - T_{i(n+1)}^{m(n+1)}x_{n+1}\| + \|T_{i(n+1)}^{m(n+1)}x_{n+1} - T_{i(n+1)}x_{n+1}\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{i(n+1)}^{m(n+1)}x_{n+1}\| + \|T_{i(n+1)}^{m(n+1)}x_{n+1} - T_{i(n+1)}x_{n+1}\| \end{aligned}$$

but,

$$\begin{aligned} \|T_{i(n+1)}^{m(n+1)-1}x_{n+1} - x_{n+1}\| &\leq \|T_{i(n+1)}^{m(n+1)-1}x_{n+1} - T_{i(n+1)}^{m(n+1)-1}m(n+1) - 1_{n+1-m}\| \\ &\quad + \|T_{i(n+1)}^{m(n+1)-1}x_{n+1-m} - x_{n+1-m}\| + \|x_{n+1-m} - x_{n+1}\| \\ &\leq (2 + k_{m(n+1)-1})\|x_{n+1} - x_{n+1-m}\| + \|T_{i(n+1-m)}^{m(n+1-m}x_{n+1-m} - x_{n+1-m}\| \\ &\quad + v_{m(n+1)-1} \end{aligned} \tag{10}$$

so that by hypothesis, (9), (8) and the uniform countinuity of $T_i\ i\ \in I$ we have:

$$\lim_{n\to\infty} \|x_n - T_{i(n+1)}x_{n+1}\| = 0 \tag{11}$$

Furthermore,

$$\|x_{n+1} - T_{i(n+1)}x_{n+1}\| \leq \|x_{n+1} - x_n\| + \|x_n - T_{i(n+1)}x_{n+1}\|. \tag{12}$$

So that using (7) and (11) we have,

$$\lim_{n\to\infty} \|x_{n+1} - T_{i(n+1)}x_{n+1}\| = 0. \tag{13}$$

Now, let $k \in I$ be arbitrarily choosen, then,

$$\|x_n - T_{i(n+k)}x_n\| \leq \|x_n - x_{n+k}\| + \|x_{n+k} - T_{i(n+k)}x_{n+k}\| + \|T_{i(n+k)}x_{n+k} - T_{i(n+k)}x_n\|$$

By uniform continuity of $T_i\ \forall\ i \in I$ and from (8) and (13) we have that

$$\lim_{n\to\infty} \|x_n - T_{i(n+k)}x_n\| = 0 \forall k \in I = \{1, ..., m\}$$

Observe that $\forall k \in I = \{1, .., m\}\ \ \exists\ \ \eta_k\ \in I$ such that $i(n) + \eta_k \equiv kmodm$, put in another way, $\forall k \in I\ \ \exists\ \ i_k\ \in I$ such that $i(n+k) \equiv imodN$ Hence,

$$\lim_{n\to\infty} \|x_n - T_ix_n\| = 0; \forall i \in \{1, ..., m\}.$$

Since $(1 - T_i)$ is demiclosed at $0 \in H\ \ \forall i$. $\{x_n\}$ is bounded and H is reflexive,
so, $\exists\ z \in K$ and $\{x_{nj}\} \subset \{x_n\}$ such that, $x_{nj} \to^w z$ as $j \to \infty$. Since, $x_{nj} - T_ix_{nj} \to 0$ as $j \to \infty\ \forall i$ then $z \in F(T_i)\ \ \forall i$ and so $z \in F = \bigcap_{i=1}^{m} F(T_i)$

Let $q \in \omega_w(x_n)$ arbitrary. Then $\exists\ \{x_{nr}\} \subset \{x_n\} \ni x_{nr} \to^w q$ and $x_{nr} - T_ix_{nr} \to 0$ as $r \to \infty\ \ \forall i$. So that since $1 - T_i$ is demiclosed at $0\ \forall i,\ \ q \in F$. Hence, $\omega_w(x_n) \subseteq F$. Moreover, $\|x_n - x_o\| \leq \|x_o - x^*\|\ \forall n \geq 0$ where $x^* = P_Fx_o$. Then by the lemma 1.2 $\{x_n\}$ converges strongly to $x^* = P_Fx_o$ (that is the common fixed point nearest to x_0).

Theorem 2.2 Let H, K, T_i, and F be as in Theorem 2.1, then $\{x_n\}_{n\geq 1}$ generated iteratively by

$$\begin{aligned} y_{n,0} &= x_n;\ y_{n,i} = (1-\alpha_n)x_n + \alpha_n T_i^n y_{n,i-1};\ i = 1, ..., m \\ K_{n,i} &= \{z \in K : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)a^i \sum_{j=0}^{i-1} \|x_n - T_{i-j}^n y_{n,i-j-1}\|^2 + \sigma_{n,i}\} \\ K_n &= \bigcap_{i=1}^{m} K_{n,i} \end{aligned} \tag{14}$$

$$\begin{aligned} Q_n &= \{z \in K : \langle x_n - z, x_o - x_n\rangle \geq 0\} \\ x_{n+1} &= P_{K_n \cap Q_n} x_0. \end{aligned} \tag{15}$$

converges to $P_F x_0$ where $\sigma_{n,i} = d_o \sum_{j=1}^{i} [(k_{n,j}^2 - 1) + v_{n,j}],\ i \in I, 0 < a \leq \alpha_n \leq b < 1.$

Proof

Let $x^* \in F$.

$$\begin{aligned} \|y_{n,i} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_{n,i}^2 \|y_{n,i-1} - x^*\|^2 \\ &\quad + \alpha_n(2k_{n,i}\|y_{n,i-1} - x^*\| + v_{n,i})\eta_{n,i} - \alpha_n(1-\alpha_n)\|x_n - T_i^n y_{n,i-1}\|^2 \end{aligned}$$

So,

$$\begin{aligned} \|y_{n,1} - x^*\|^2 &\leq \big(1 + \alpha_n(k_{n,1}^2 - 1)\big)\|x_n - x^*\|^2 + \alpha_n(2k_{n,1}\|x_n - x^*\| + v_{n,1})v_{n,1} \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - T_1^n x_n\| \end{aligned} \tag{16}$$

So,

$$\begin{aligned} \|y_{n,i} - x^*\|^2 &\leq \left(1 + \sum_{j=0}^{i-1} \alpha_n^{j+1} \Pi_{t=0}^{j-1} k_{n,i-t}^2 (k_{n,i-j}^2 - 1)\right)\|x_n - x^*\|^2 \\ &\quad + \sum_{j=0}^{i-1} \alpha_n^{j+1}(2k_{n,i-j}\|y_{n,i-j-1} - x^*\|^2 + v_{n,i-j})v_{n,i-j}\Pi_{t=0}^{j-1} k_{n,i-t}^2 \\ &\quad - \sum_{j=0}^{j-1} \alpha_n^{j+1}(1-\alpha_n)\|x_n - T_{i-j}^n y_{n,i-j-1}\|^2 \Pi_{t=0}^{j-1} k_{n,i-t}^2 \\ &\leq \big(1 + bq^i \sum_{j=1}^{i}(k_{n,j}^2 - 1)\big)\|x_n - x^*\|^2 \\ &\quad + bq^i d_1 \sum_{j=1}^{i} v_{n,j} - a^i(1-b)\sum_{j=1}^{i}\|x_n - T_j^n y_{n,j-1}\|^2 \\ &\leq \|x_n - x^*\|^2 - a^i(1-b)\sum_{j=1}^{i}\|x_n - T_j^n y_{n,j-1}\|^2 + \sigma_{n,i} \end{aligned} \tag{17}$$

where $d_o = bq^i \max\{d_1, (diamK)^2\}$ so, $x^* \in K_{n,i} \forall\ i$ and hence $x^* \in \bigcap_{i=1}^{m} K_{n,i}\ \forall\ n$. So, $F \subset K_{n,i} \forall\ n \geq 0,\ \forall\ i$ and hence, $F \subset K_n$. Following the argument as in Theorem 2.1, we have that $F \subset Q_n\ \forall n \geq 0$ and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Now,

$$\alpha_n^2 \|x_n - T_i^n y_{n,i-1}\|^2 \leq \|y_{n,i} - x_{n+1}\|^2 + 2\|y_{n,i} - x_{n+1}\|.\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2$$

but $x_{n+1} \in K_n$, so, $\forall i \in I$

$$\|y_{n,i} - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_n(1-\alpha_n)a^i \sum_{j=1}^{i} \|x_n - T_j^n y_{n,j-1}\|^2 + \sigma_{n,i}.$$

so that $\forall i \in I$

$$\alpha_n^2\|x_n - T_i^n y_{n,i-1}\|^2 \leq \|x_{n+1} - x_n\|^2 + \sigma_{n,i} - \alpha_n(1-\alpha_n)a^i \sum_{j=1}^{i} \|x_n - T_j^n y_{n,j-1}\|^2 + 2\|y_{n,i} - x_{n+1}\|.\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2$$

and so $\forall i \in I$

$$\begin{aligned} a^2\|x_n - T_i^n y_{n,i-1}\|^2 &\leq \alpha_n^2\|x_n - T_i^n y_{n,i-1}\|^2 \\ &\leq 2\|x_{n+1} - x_n\|^2 + \sigma_{n,i} + 2\|y_{n,i} - x_{n+1}\|.\|x_{n+1} - x_n\| \end{aligned}$$

Hence, $\lim_{n\to\infty} \|x_n - T_i^n y_{n,i-1}\| = 0 \ \forall i \in I$ and hence, $\lim_{n\to\infty} \|y_{n,i} - x_n\| = 0 \ \forall i \in I$

$$\|x_n - T_i^n x_n\| \leq \|x_n - T_i^n y_{n,i-1}\| + k_{n,i}\|y_{n,i-1} - x_n\| + v_{n,i}$$

Thus, $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0 \ \forall i \in I$.

$$\|x_n - T_i x_n\| \leq \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i x_n\|$$

$$\|x_n - T_i^{n-1} x_n\| \leq (1 + k_{n-1,i})\|x_n - x_{n-1}\| + \|x_{n-1} - T_i^{n-1} x_{n-1}\|$$

Thus, $\lim_{n\to\infty} \|x_n - T_i^{n-1} x_n\| = 0 \ \forall i \in I$, so that by uniform continuity of $T_i \ \forall i \in I$, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$. Now, since $(1 - T_i)$ is demiclosed at 0, the same argument in Theorem 2.1 completes the Proof.

3. Conclusion

CQ Algorithms for iterative approximation of a common fixed point of a finite family of nonlinear maps were introduced and sufficient conditions for the strong convergence of this process to a common fixed point of the family of Total asymptotically Nonexpansive maps were proved. Our iterative processes generalise some of the existing ones, our theorems improve, generalise and extend several known results and our method of proof is of independent interest.

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Conflict of Interest

Authors declare that they do not have any conflict of interest.

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