

# Approximate Method to Compute Hypersingular Finite-Part Integrals with Rapidly Oscillating Kernels

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**Abstract** — In this paper, an algorithm for the numerical evaluation of hypersingular finite-part integrals with rapidly oscillating kernels is proposed. The method is based on an interpolatory procedure at zeros of the orthogonal polynomials with respect to the first kind Chebyshev weight. Bounds of the error and of the amplification factor are also provided. Numerically stable procedure are obtained and the corresponding algorithms can be implemented in a fast way.

**Keywords** — Error bound, Finite-part integral, Oscillatory integral, Stability.

## I. INTRODUCTION

The mathematical modeling of many areas of physics and technology, such as fracture mechanics, electromagnetic scattering (see [1-4] and the references therein) give rise to importance of evaluating singular and hypersingular integrals with rapidly oscillating kernel.

Recently, approximate method for evaluating Cauchy singular integral of the form:

$$I^{(1)}(\omega, f; t) := \int_{-1}^1 \frac{f(x)}{(x-t)} e^{i\omega x} dx, \omega > 0, -1 < t < 1, \quad (1)$$

is proposed in [5], where one can be found the comparison with the very few other methods available in the previous literature.

In this paper, we propose a method to approximate the finite-part hypersingular integral:

$$I^{(2)}(\omega, f; t) := \int_{-1}^1 \frac{f(x)}{(x-t)^2} e^{i\omega x} dx, \omega > 0, -1 < t < 1, \quad (2)$$

taking into account the results in [5]. Only a small number of publications deals with the numerical evaluation of integrals in (2) (see [6,7]). In particular, the authors of [6] place their attention to construct of an optional with respect the order passive algorithms for evaluating Cauchy singular and hypersingular integrals with oscillating kernel as in (1) and (2). The idea followed here is entirely different consisting in proposing quadrature formulas of interpolation type that are convergent and stable with little computational effort.

Defined the integral (2) as Cauchy principal value:

$$I^{(2)}(\omega, f; t) = \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^1 \right\} \frac{f(x)}{(x-t)} e^{i\omega x} dx, \quad \omega > 0, -1 < t < 1,$$

the integral (2) can be transferred in the derivative of (1):

$$I^{(2)}(\omega, f; t) = \frac{d}{dt} I^{(1)}(\omega, f; t), \omega > 0, -1 < t < 1, \quad (3)$$

see [8]. Further,

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$$I^{(2)}(\omega, f; t) = \int_{-1}^1 \frac{d}{dx} \left( \frac{f(x)e^{i\omega x}}{x-t} \right) dx + \frac{e^{i\omega} f(1)}{1-x} + \frac{e^{-i\omega} f(-1)}{1+x}, \quad \omega > 0, -1 < t < 1, \quad (4)$$

(see Ch. II, Lemma 6.1 in [9]). Before proceeding further, we recall that the existence of (1) is ensured, by assuming that the function  $f$  is Hölder continuous while the existence of the Hölder continuous derivative  $f'$  assures the existence of integral in (2), (cf. [8]).

In the next section, we present a method for evaluating integral in (2) making use of the values of  $f$  and  $f'$  at the first kind Chebyshev zeros, proving bounds of the error and of the amplification factor. Although the deduced formula has the drawback of using twice as many functional evaluations, it has the advantage of having recourse only to the weights of the quadrature sum proposed in [5] to approximate (1).

Bounds of the error and of the amplification factor are provided.

## II. A METHOD TO COMPUTE INTEGRAL (2)

We start remarking that since (4) the numerical approximation of (2) can be connected to the quadrature of integrals of the following kind:

$$I^\omega(f; t) := \int_{-1}^1 \frac{f(x)}{x-t} \sin \omega x dx, \quad J^\omega(f; t) := \int_{-1}^1 \frac{f(x)}{x-t} \cos \omega x dx, \quad (5)$$

and  $I^\omega(f'; t)$ ,  $J^\omega(f'; t)$ .

The approximation of the integrals in (5) can be obtained by using the methods proposed and studied in [5]. Therefore in the following we will focus on quadrature of  $I^\omega(f'; t)$ , also because the integral  $J^\omega(f'; t)$  can be treated in entirely analogous way. Two different methods can be followed approximating  $I^\omega(f'; t)$  by

$$I_m^\omega(v^{\alpha, \beta}; f; t) := \int_{-1}^1 \frac{L_m(v^{\alpha, \beta}; f'; t)}{x-t} \sin \omega x dx, \quad (6)$$

Or

$$\widehat{I}_m^\omega(v^{\alpha, \beta}; f; t) := \int_{-1}^1 \frac{L'_m(v^{\alpha, \beta}; f; t)}{x-t} \sin \omega x dx, \quad (7)$$

where the Lagrange polynomial  $L_m(v^{\alpha, \beta}; g)$  interpolates a given function  $g$  at the points  $x_{m,k}^{\alpha, \beta}$ ,  $k = 1, \dots, m$  zeros of the  $m$ th Jacobi polynomial  $p_m^{\alpha, \beta}$ ,  $m \in \mathbb{N}$ , with respect to the exponent  $\alpha, \beta > -1$ .

In the present section we focus our attention on the use of (6) to approximate  $I^\omega(f'; t)$  while in a further work [10] we propose and study the approximation of the same integral by (7). The major difference consists in the fact that if (6) is adopted it follows the overall approximation of (2) requires the computation of the function values  $f(x_{m,k}^{\alpha, \beta})$  and  $f'(x_{m,k}^{\alpha, \beta})$ ,  $k = 1, \dots, m$ , while the use of (7) requires only the evaluations  $f(x_{m,k}^{\alpha, \beta})$ ,  $k = 1, \dots, m$ . Therefore the (7) leads to a lower computational cost, in addition to the fact that its use appears essential in quadrature methods for solving hypersingular integral equations. Nevertheless, the formula presented here has the advantage of having recourse only to the weights of the quadrature sum proposed into approximate (1).

In the next we study the convergence of (6) to  $I^\omega(f'; t)$ . Let us denote by  $\omega_\varphi(f; \delta)$  the modulus of smoothness of a given function  $g$ , defined as:

$$\omega_{-\varphi} := \sup_{h \leq \delta} \max_{|x| \leq 1} |\Delta_{h\varphi} g(x)|,$$

where  $\varphi(x) = \sqrt{1-x^2}$  and  $\Delta_{h\varphi} g(x) = g\left(x + \frac{h}{2}\varphi(x)\right) - g\left(x - \frac{h}{2}\varphi(x)\right)$ , (cf. [11]). Further, we denote by  $\|g\|_\infty = \max_{|x| \leq 1} |g(x)|$  the usual uniform norm and by  $\Lambda_m^{\alpha, \beta}$ ,  $m \in \mathbb{N}$  the  $m$ th Lebesgue constant corresponding to the weight function  $v^{\alpha, \beta}$ . Then, for the quadrature (6) the next result holds true.

**Theorem 1.** For every function  $f \in C^1$  and  $\omega \geq 0$  we have

$$|I_m^\omega(v^{\alpha, \beta}; f; t)| \leq C \log \frac{e}{1-t^2} (1 + \omega + \log m) \Lambda_m^{\alpha, \beta} |f'|_\infty, \quad (8)$$

and

$$|I^\omega(f'; t) - I_m^\omega(v^{\alpha, \beta}; f; t)| \leq C \log \frac{e}{1-t^2} \cdot \left\{ (1 + \omega + \log m) \Lambda_m^{\alpha, \beta} \omega_\varphi \left( f'; \frac{1}{m} \right) + \int_0^{1/m} \frac{\omega_\varphi(f'; \delta)}{\delta} d\delta \right\},$$

where  $C$  denotes a positive constant independent of  $m, f, \omega$  and  $t \in (-1, 1)$ .

**Proof.** See Theorem 3.2 in [5].

We remark that by (8) we can know the behavior of the weighted amplification factor. Indeed, we deduce the following bound

$$\left\| I_m^\omega(v^{\alpha, \beta} f) \log^{-1} \frac{e}{1-(\cdot)^2} \right\| \leq C(1 + \omega + \log m) \|f'\|_\infty \Lambda_m^{\alpha, \beta},$$

where  $C$  denotes a positive constant independent of  $m, f$  and  $\omega$ .

Now let us see how to compute (6). For simplicity we assume  $\alpha, \beta = -1/2$  even if the generalization to any value of  $\alpha, \beta > -1$  is really trivial. So we denote by  $T_m = p_m^{-1/2, -1/2}$  the Chebyshev orthogonal polynomial of the first kind and let  $x_{m,k} = x_{m,k}^{-1/2, -1/2}$  be the zeros of  $T_m$ . Since,

$$\mathcal{L}_m(v^{-1/2, -1/2}; f'; x) = \mathcal{L}_m(f'; x) = \sum_{k=1}^m \ell_{m,k}(x) f'(x_{m,k}),$$

where  $\ell_{m,k}, k = 1, \dots, m$  are the fundamental Lagrange polynomials with respect to the points  $x_{m,k}, k = 1, \dots, m$ , we have:

$$I_m^\omega(v^{-1/2, -1/2}; f; t) = I_m^\omega(f; t) = \sum_{k=1}^m \left[ \int_{-1}^1 \frac{\ell_{m,k}(x)}{x-t} \sin \omega x dx \right] f'(x_{m,k}),$$

where

$$\ell_{m,k}(x) = \sum_{i=0}^{m-1} a_i T_i(x), \quad k = 1, \dots, m,$$

with

$$a_i = \int_{-1}^1 \ell_{m,k}(x) T_i(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{m} \sum_{j=1}^m \ell_{m,k}(x_{m,j}) T_i(x_{m,j}) = \frac{\pi}{m} T_i(x_{m,k}), i = 1, \dots, m-1.$$

So,

$$I_m^\omega(f; t) = \frac{\pi}{m} \sum_{k=1}^m \left[ \sum_{i=0}^{m-1} T_i(x_{m,k}) q_i^\omega(t) \right] f'(x_{m,k}), \quad (9)$$

where

$$q_i^\omega(t) = \int_{-1}^1 \frac{T_i(x)}{x-t} \sin \omega x dx, \quad i = 0, 1, \dots$$

Recalling that the polynomials  $T_n, n \in \mathbb{N}$  satisfy:

$$T_0(x) \equiv 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \dots,$$

we try

$$q_{n+1}^\omega(t) = 2tq_n^\omega(t) - q_{n-1}^\omega(t) + 2M_n^\omega, \quad n = 1, 2, \dots, \quad (10)$$

where

$$M_n^\omega = \int_{-1}^1 T_n(x) \sin \omega x dx, \quad n = 0, 1, \dots$$

The accurate evaluation of the integral  $M_n^\omega$  in (10) allows us to compute  $q_n^\omega(t)$  for  $n = 1, 2, \dots$ , together with:

$$q_0^\omega(t) = \int_{-1}^1 \frac{\sin \omega x}{x-t} dx = \sin \omega x [\text{Ci}(\tau_1) - \text{Ci}(|\tau_2|)] + \cos \omega x [\text{Si}(\tau_1) + \text{Si}(|\tau_2|)],$$

and

$$q_1^\omega(t) = \int_{-1}^1 \frac{x \sin \omega x}{x-t} dx = t q_0^\omega(t),$$

where

$$\text{Si}(\tau) = \int_0^\tau \frac{\sin x}{x} dx, \quad \text{Ci}(\tau) = \int_0^\tau \frac{\cos x - 1}{x} dx + \log \tau + C, \quad \tau > 0,$$

are the sine and cosine integral, respectively;  $\tau_1 = \omega(1-t)$ ,  $\tau_2 = -\omega(1+t)$  and  $C$  is the Euler constant. The starting values of (10) require the evaluation of the sine and cosine integrals that can be computed by some mathematical software like Mathematica [12].

Finally, we remark that (9) can be rewritten:

$$I_m^\omega(f; t) = \frac{\pi}{m} \sum_{i=0}^{m-1} A_{m,i}(f) q_i^\omega(t) \quad (11)$$

with

$$A_{m,i}(f) = \sum_{k=1}^m T_i(x_{m,k}) f'(x_{m,k}), \quad i = 0, \dots, m-1.$$

The coefficients  $A_{m,i}(f)$ ,  $i = 0, \dots, m-1$ , are not influenced by the value  $t$  and the oscillatory factor  $\omega$ . Thus the evaluation of  $I_m^\omega(f; t)$  in (11) can be done following Clenshaw type algorithm:

$$\begin{aligned} z_{m+1} &= z_m = 0, & w_m &= 0, \\ z_k &= 2tz_{k+1} - z_{k+2} + A_{m,k}(f), & k &= m-1, m-2, \dots, 0, \\ w_k &= 2z_{k+1}M_k^\omega + w_{k+1}, \\ I_m^\omega(f; t) &= q_0^\omega(t)z_0 + w_0. \end{aligned}$$

#### CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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