Suzuki-Type of Common Fixed Point Theorems in S-Fuzzy Metric Spaces

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***Abstract* — In this paper, by using of Suzuki-type approach [Suzuki, T., A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136, 1861–1869, 2008.] we prove new type of Suzuki- type fixed point theorem for non-Archimedean S - fuzzy metric spaces which is generalization of Suzuki-Type fixed point results in S - metric spaces.**

***Key words* — Fixed point, Suzuki-Contraction, S-fuzzy metric space.**

I. Introduction

In 1965, the concept of the fuzzy set was initially investigated by Zadeh [13]. Then in 1975, Kramosil and Michalek [4] introduced the fuzzy metric space as a generalization of a metric space. In 1994, George and Veeramani [2] modified the notion of fuzzy metric spaces by using continuous t-norms. Fixed point theorems for contractive mappings in metric spaces have been studied by many authors (see [1], [5], [12]). The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. In 2008 Suzuki [6] introduced an interesting generalization of Banach contraction principle. Recently, Sedghi, Shobe and Aliouche [8] have defined the concept of S-metric space as a generalization of a metric space, (see [3], [7], [9]-[11]) and proved some fixed point results. In this paper, we introduce the new contractive condition in the frame work of non-Archimedean S-fuzzy metric spaces. We also prove the corresponding coincidence fixed point theorem for two mappings in this framework.

II. Preliminaries

A. *Definition 2.1*

A binary operation $*$: $[0, 1] \times [0,1] \rightarrow [0, 1]$ is a continuous t – norm if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$, for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Examples for continuous t – norm are $a \times b = min\{a, b\}$ and $a \times b = ab$.

B. *Definition 2.2*

A 3 – tuple (X, S, $*$) is called S-fuzzy metric space if X is an arbitrary non – empty set, $*$ is a continuous t – norm, and S is a fuzzy set on $X^3 \times [0, \infty)$ satisfying the following conditions. For each $x, y, z, a \in X$ and $r, s, t > 0$:

1. $S(x, y, z, t) > 0$;
2. $S(x, y, z, t) = 1$ *if* $x = y = z$;
3. $S(x, y, z, t = S(p\{x, y, z\}, t)$, where p is a permutation;
4. $S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t) \leq S(x, y, z, r + s + t)$, (Tetrahedral inequality) .
5. $S(x, y, z, .) : (0, \infty) \rightarrow [0, 1]$ is continuous.

C. *Example 2.3*

Let X = [0, 1] with a usual metric. Define

$$S(x, y, z, t) = min\{M(x, y, t), M(y, z, t), M(z, x, t)\}$$

where $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, S,$*$) is called S –fuzzy metric space.

D. *Definition 2.4*

A 3 – tuple (X, S, $*$) is called non-Archimedean S-fuzzy metric space if X is an arbitrary non – empty set, $*$ is a continuous t – norm, and S is a fuzzy set on $X^3 \times [0, \infty)$ satisfying the following conditions. For each $x, y, z, w \in X$ and $r, t, u > 0$:

1. $S(x, y, z, t) > 0$,
2. $S(x, y, z, t) = 1$ *if* $x = y = z$,
3. $S(x, y, z, t) = S(p\{x, y, z\}, t)$, where p is a permutation
4. $S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$

$$\leq S(x, y, z, r \vee t \vee u),$$

where $r \vee t \vee u$= max $\{r, t, u\}$, (Tetrahedral inequality) .

5. $S(x, y, z, .) : (0, \infty) \rightarrow [0, 1]$ is continuous.

E. *Definition 2.5*

Let (X, S, $*$) be a S- fuzzy Metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

1. A sequence $\{x_n\}$ is said to be convergent to x if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.
2. A sequence $\{x_n\}$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and there exists $n_0 \in \mathbb{N}$ such that $S(x_m, x_n, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

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3. The S-fuzzy metric space is called complete if every Cauchy sequence is convergent.

F. Lemma 2. 6

Let (X, S, $*$) be an S-fuzzy metric space. If there exists sequence $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n, t) = S(x,x,y,t)$ for all x, $y \in X$.

III. Main Results

A. Theorem 3.1

Let $a * b = ab$ for all a, b$\in[0, 1]$ and (X, S, $*$) be a complete non- Archimedean S - fuzzy metric space. Let T, R: X $\to$ X be two self maps and $\theta: [0,1) \to (\frac{1}{2}, 1]$ be defined by:

$$\theta(r) = \begin{cases} 1, 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2}, \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}} \\ \frac{1}{1+r}, \frac{1}{\sqrt{2}} \le r \le 1 \end{cases} \qquad (1)$$

If there exists $r \in [0,1)$ such that for each $x, y, z \in X$, $t > 0$ satisfying the condition:

$$\max\{S(x,y,Tx,t), S(x,y,Rx,t)\}^{\theta(r)} \ge S(x,y,z,t) \quad (2)$$

implies

$$\begin{Bmatrix} S(Rx,Ry,Rz,t) * S(Tx,Ty,Tz,t) * \\ S(Rx,Ry,Tz,t) * S(Tx,Ty,Rz,t) \end{Bmatrix} \ge S(x,y,z,t)^r \quad (3)$$

Then there exists a unique common fixed point of T and R.

B. Proof

At first, we prove that if v is a fixed point of T, then it is also fixed point of R and vice versa. Hence, let v be a fixed point of T, then we will show that $Rv = v$.

Taking $x, y = v$ *and* $z = Tv$ in (2) and (3), We get:

$$1 = max\{S(v,v,Tv,t), S(v,v,Rv,t)\}^{\theta(r)} > S(v,v,Tv,t)$$

implies

$$\begin{Bmatrix} S(Rv,Rv,RTv,t) * S(Tv,Tv,T^2v,t) \\ * S(Rv,Rv,T^2v,t) * S(RTv,Tv,Tv,t) \end{Bmatrix} \ge S(v,v,Tv,t)^r = 1 \quad (4)$$

Thus, $S(Rv,Rv,v,t) = 1$. i.e., v is a fixed point of R. Similarly, we can show that if v is a fixed point of R, then it is also fixed point of T. Now to prove our theorem, it is enough to prove that T and R has a fixed point.

Putting $x, y = x$ and $z = Tx$ in (2) and (3), we get:

$$\max\{S(x,x,Tx,t), S(x,x,Rx,t)\}^{\theta(r)} \ge S(x,x,Tx,t)$$

implies

$$\begin{Bmatrix} S(Rx,Rx,RTx,t) * S(Tx,Tx,T^2x,t) * \\ S(Rx,Rx,T^2x,t) * S(RTx,Tx,Tx,t) \end{Bmatrix} \ge S(x,x,Tx,t)^r$$

Hence,

$$S(Tx,Tx,T^2x,t) \ge S(x,x,Tx,t)^r \text{ for all } x \in X \quad (5)$$

and

$$S(Tx,Tx,RTx,t) \ge S(x,x,Tx,t) \text{ for all } x \in X. \quad (6)$$

Putting $x, y = x$ and $z = Rx$ in (2.2) and (2.3). Hence, form

$$max\{S(x,x,Tx,t), S(x,x,Rx,t)\}^{\theta(r)} \ge S(x,x,Rx,t)$$

implies

$$\begin{Bmatrix} S(Rx,Rx,R^2x,t) * S(Tx,Ty,TRx,t) * \\ S(Rx,Rx,TRx,t) * S(R^2x,Tx,Tx,t) \end{Bmatrix} \ge S(x,x,Rx,t)^r$$

Hence,

$$S(Rx,Rx,TRx,t) \ge S(x,x,Rx,t)^r \text{ for all } x \in X \quad (7)$$

Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1 = Rx_0$ and $x_{2n+1} = Rx_{2n}, x_{2n+2} = Tx_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. By (6), we have

$$S(x_{2n}, x_{2n}, x_{2n+1}, t) = S(Tx_{2n-1}, Tx_{2n-1}, RTx_{2n-1}, t)$$

$$\ge S(x_{2n-1}, x_{2n-1}, Tx_{2n-1}, t)^r$$

$$= S(x_{2n-1}, x_{2n-1}, x_{2n}, t)^r. \quad (8)$$

Also, by (7) we have:

$$S(x_{2n+1}, x_{2n+1}, x_{2n+2}, t) = S(Rx_{2n}, Rx_{2n}, TRx_{2n}, t) \ge S(x_{2n}, x_{2n}, Rx_{2n}, t)^r$$

$$= S(x_{2n}, x_{2n}, x_{2n+1}, t)^r. \quad (9)$$

By (8), (9) we have:

$$S(x_n, x_n, x_{n+1}, t) \ge S(x_{n-1}, x_{n-1}, x_n, t)^r. \quad (10)$$

Hence, by induction, we have:

$$S(x_n, x_n, x_{n+1}, t) \ge S(x_{n_1}, x_{n_1}, x_n, t)^r \ge \cdots \ge S(x_0, x_0, x_1, t)^{r^n} \quad (11)$$

By Tetrahedral inequality in S – fuzzy metric space for m $> n$ we have,

$$S(x_n, x_n, x_m, t) \ge S(x_n, x_n, x_{n+1}, t) * S(x_{n+1}, x_{n+1}, x_{n+2}, t) * \cdots * S(x_{m-1}, x_{m-1}, x_m, t)$$

$$\ge S(x_0, x_0, x_1, t)^{r^n} * S(x_0, x_0, x_1, t)^{r^{n+1}}$$

$$* \cdots * S(x_0, x_0, x_1, t)^{r^{m-1}}$$

$$\geq S(x_0, x_0, x_1, t)^{r^n} . S(x_0, x_0, x_1, t)^{r^{n+1}} .$$

$$\cdots . S(x_0, x_0, x_1, t)^{r^{m-1}}$$

As $0 < r < 1$, we have:

$$S(x_n, x_n, x_m, t) = S(x_0, x_0, x_1, t)^{r^n(1+r+r^2+\cdots+r^{m-n-1})}$$

$$\geq S(x_0, x_0, x_1, t)^{\frac{r^m}{1-r}} \to 1 \text{ as m } \to \infty.$$

So, we have $\lim_{n,m \to\infty} S(x_n, x_n, x_m, t) = 1.$

It follows that $\{x_n\}$ is a Cauchy sequence. Since X is Complete S- fuzzy metric space, there is some v in X such that,

$$\lim_{n\to\infty} Rx_{2n} = \lim_{n\to\infty} x_{2n+1} = v$$

and

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = v$$

We show that v is a common fixed point of T and R.

It is enough to prove that Tv = v. Now we consider x $\in$ X with x $\neq$ v.

As $\lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t) = 1$ and $\lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, x, t) \neq 1$, therefore there exists some $x_{2n_k+1} \in$ X such that

$$\max\begin{Bmatrix} S(x_{2n_k+1}, x_{2n_k+1}, Tx_{2n_k+1}, t), \\ S(x_{2n_k+1}, x_{2n_k+1}, Rx_{2n_k+1}, t) \end{Bmatrix}^{\theta(r)}$$

$$\geq S(x_{2n_k+1}, x_{2n_k+1}, x, t)$$

implies,

$$\begin{Bmatrix} S(Rx_{2n_k+1}, Rx_{2n_k+1}, Rx, t) * S(Tx_{2n_k+1}, Tx_{2n_k+1}, Tx, t) \\ * S(Rx_{2n_k+1}, Rx_{2n_k+1}, Tx_{2n_k+1}, t) * S(Rx, Tx_{2n_k+1}, Tx_{2n_k+1}, t) \end{Bmatrix} \geq S(x_{2n_k+1}, x_{2n_k+1}, x, t)^r$$

Hence,

$$S(x_{2n_k+2}, x_{2n_k+2}, Tx, t) = S(Tx_{2n_k+1}, Tx_{2n_k+1}, Tx, t) \geq S(x_{2n_k+1}, x_{2n_k+1}, x, t)^r$$

Taking limit as $n \to \infty$, we have:

$$S(v, v, Tx, t) = \lim_{n\to\infty} S(x_{2n_k+2}, x_{2n_k+2}, Tx, t)$$

$$\geq \lim_{n\to\infty} S(x_{2n_k+1}, x_{2n_k+1}, x, t)^r$$

$$= S(v, v, x, t)^r$$

Therefore, for each $x \neq v$,

$$S(v, v, Tx, t) \geq S(v, v, x, t)^r \tag{12}$$

Now by induction, we prove that,

$S(v, v, T^n v, t) \geq S(v, v, Tv, t)$ for all $n \in \mathbb{N}$. (13)

For n = 1, the inequality is obvious.

Suppose the inequality (13) is true for some $m \in \mathbb{N}$

$$S(v, v, T^m v, t) \geq S(v, v, Tv, t)$$

Now for $n = m + 1$, if $T^m v = v$ then

$$S(v, v, T^{m+1} v, t) = S(v, v, Tv, t)$$

If $T^m v \neq v$ then by (12)

$$S(v, v, T^{m+1} v, t) \geq S(v, v, T^m v, t)^r$$

$$\geq S(v, v, Tv, t)^r$$

$$> S(v, v, Tv, t) \tag{14}$$

Hence, the inequality (13) holds for each n $\in \mathbb{N}$.

Let us assume that Tv $\neq$ v. Now, we prove the following inequality by the principle of mathematical induction $S(Tv, Tv, T^n v, t) \geq S(v, v, Tv, t)^r$ for each $n \in \mathbb{N}$ (15)

For n = 1, it is obvious. Further from (5) the inequality (15) holds for n = 2.

Suppose (15) holds for some n > 2, then we have:

$$S(Tv, Tv, v, t) = S(v, v, Tv, t)$$

$$\geq S(v, v, T^n v, t) * 2S(Tv, Tv, T^n v, t)$$

$$\geq S(v, v, T^n v, t) * 2S(v, v, Tv, t)^r$$

Therefore,

$$S(v, v, Tv, t)^{(1-r)} \geq S(v, v, T^n v, t). \tag{16}$$

Case I: $0 \leq r \leq \frac{1}{\sqrt{2}}$ (hence $\theta(r) = \frac{1-r}{r^2}$) Hence,

$$\max\{S(T^n v, T^n v, T^{n+1} v, t), S(T^n v, T^n v, RT^n v, t)\}^{\theta(r)}$$

$$\geq S(T^n v, T^n v, T^{n+1} v, t)^{\frac{1-r}{r^2}}$$

$$\geq S(T^n v, T^n v, T^{n+1} v, t)^{\frac{1-r}{r^2}}$$

So, by (16), we have:

$$\max\{S(T^n v, T^n v, T^{n+1} v, t), S(T^n v, T^n v, RT^n v, t)\}^{\theta(r)}$$

$$\geq S(T^n v, T^n v, T^{n+1} v, t)^{\frac{1-r}{r^2}}$$

$$\geq S(v, v, Tv, t)^{1-r}$$

$$\geq S(v, v, T^n v, t)$$

$$= S(T^n v, T^n v, v, t),$$

implies that

$$\left\{\begin{matrix} S(RT^n v, RT^n v, Rv, t) * S(T^{n+1}v, T^{n+1}v, Tv, t) * \\ S(RT^n v, RT^n v, Tv, t) * S(Rv, T^{n+1}v, T^{n+1}v, t) \end{matrix}\right\} \geq S(T^n v, T^n v, v, t)^r$$

Using (10), we obtain

$$S(Tv, Tv, T^{n+1}v, t) \geq S(v, v, T^n v, t)^r$$

$$\geq S(v, v, Tv, t)^r$$

So, the inequality (14) holds for eah $n \in \mathbb{N}$. Now $\mathrm{Tv} \neq \mathrm{v}$ and (14) implies that $\mathrm{T^n v} \neq \mathrm{v}$ (If $T^n v = v$, then we find $S(T^n v, T^n v, v, t) \geq S(v, v, Tv, t)^r$ implies:

$$S(v, v, Tv, t) \leq S(v, v, Tv, t)^r$$

$< S(v, v, Tv, t)$, which is not possible.
Hence, (13) implies that:

$$S(v, v, T^{n+1}v, t) \geq S(v, v, T^n v, t)^r$$

$$\geq S(v, v, T^{n-1}v, t)^{r^2}$$

$$\geq S(v, v, Tv, t)^{r^n}$$

Hence, $\lim_{n\to\infty} \mathrm{S}(\mathrm{v}, \mathrm{v}, \mathrm{T}^{n+1}\mathrm{v}, \mathrm{t}) = 1$ this implies that $\mathrm{T}^n \to \mathrm{v}$. From this and (3.1.15), we have:

$$S(Tv, Tv, v, t) = \lim_{n\to\infty} S(Tv, Tv, T^n v, t)$$

$$\geq \lim_{n\to\infty} S(v, v, Tv, t)^r$$

$$= S(Tv, T v, v, t)^r.$$

Thus, $S(Tv, Tv, v, t) = 1$. Which is contrary to our assumption. Hence, $Tv = v$. As already proved v is fixed point of R also. Hence, $Tv = Rv = v$.

Case II: $\frac{1}{\sqrt{2}} \leq r < 1$ (hence $\theta(r) = \frac{1}{1+r}$) . We will prove that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that:

$$S(x_{n_k}, x_{n_k}, Tx_{n_k}, t)^{\frac{1}{1+r}} \geq S(x_{n_k}, x_{n_k}, v, t),$$

$$\text{or } S(x_{n_k}, x_{n_k}, Rx_{n_k}, t)^{\frac{1}{1+r}} \geq S(x_{n_k}, x_{n_k}, v, t),$$

holds for each $k \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$. Then by (10) we have,

$$\max\{S(x_n, x_n, Tx_n, t), S(x_n, x_n, Rx_n, t)\}^{\frac{1}{1+r}}$$

$$< S(x_n, x_n, v, t)$$

Hence,

$$S(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)^{\frac{1}{1+r}} < S(x_{2n+1}, x_{2n+1}, v, t)$$

and $S(x_{2n}, x_{2n}, Rx_{2n}, t) < S(x_{2n}, x_{2n}, v, t)$ holds for every $n \in \mathbb{N}$.

Then by (10) we have:

$$S(x_{2n+1}, x_{2n+1}, x_{2n}, t) \geq S(x_{2n+1}, x_{2n+1}, v, t) * S(x_{2n}, x_{2n}, v, t)$$

$$> S(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)^{\frac{1}{1+r}}$$

$$* S(x_{2n}, x_{2n}, Tx_{2n}, t)^{\frac{1}{1+r}}$$

$$> S(x_n, x_n, x_{2n+1}, t)^{\frac{r}{1+r}} * \ * S(x_{2n}, x_{2n}, x_{2n+1}, t)^{\frac{1}{1+r}}$$

$$= S(x_{2n}, x_{2n}, x_{2n+1}, t)$$

which is impossible. Hence one of the following holds for each n:

$$S(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)^{\theta(r)} \geq S(x_{2n+1}, x_{2n+1}, v, t)$$

Or

$$S(x_{2n}, x_{2n}, Rx_{2n}, t)^{\theta(r)} \geq S(x_{2n}, x_{2n}, v, t)$$

If the following inequality hold:

$$S(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)^{\theta(r)} \geq S(x_{2n+1}, x_{2n+1}, v, t)$$

implies that

$$\left\{\begin{matrix} S(Rx_{2n}, Rx_{2n}, Rv, t) * S(Tx_{2n}, Tx_{2n}, Tv, t) \\ * S(Rx_{2n}, Rx_{2n}, Tv, t) * S(Rv, Tx_{2n}, Tx_{2n}, t) \end{matrix}\right\}$$

$$\geq S(x_{2n}, x_{2n}, v, t)^r$$

Hence,

$$S(x_{2n+1}, x_{2n+1}, Tv, t) = S(Rx_{2n}, Rx_{2n}, Tv, t)$$

$$\geq S(x_{2n}, x_{2n}, v, t)^r$$

Passing to the limit when $n \to \infty$, we get that $S(v, v, Tv, t) \geq 1$, which is possible only if $Tv = v$. Similarly, if the following inequality hold.

$$S(x_{2n}, x_{2n}, Rx_{2n}, t)^{\theta(r)} \geq S(x_{2n}, x_{2n}, v, t)$$

We have $Tv = v$. Thus, we have proved that v is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Let v and u be to common fixed points of T and R, such that $v \neq u$. Taking $x, y = v$ and $z = u$ in (2) and (3), we get:

$$1 = \max\{S(v, v, Tv, t), S(v, v, Rv, t)\}^{\theta(r)} \geq S(v, v, u, t)$$

implies

$$\left\{\begin{matrix} S(Rv, Rv, Ru, t) * S(Tv, Tv, Tu, t) \\ * S(Rv, Rv, Tu, t) * S(Ru, Tv, Tv, t) \end{matrix}\right\} \geq S(v, v, u, t)^r$$

Therefore, $S(v, v, u, t) \geq S(v, v, u, t)^r > S(v, v, u, t)$, which is not possible. Hence $v = u$. This completes the proof. Taking $R = T$, we get the Suzuki type result.

C. Corollary 3.2

Let $(X, S,*)$ be a complete S-fuzzy metric space. If $T: X \to X$ be a self mapping and $\theta : [0, 1) \to (\frac{1}{2}, 1]$ is defined by (1). Assume that there exists $r \in [0, 1)$such that for each $x, y, z \in X$,

$$S(x, y, Tx, t)^{\theta(r)} \geq S(x, y, z, t)$$

implies

$$S(Tx, Ty, Tz, t) \geq S(x, y, z, t)^r$$

Then T has a unique fixed point $v \in X$.

D. Proof

To prove the corollary, It is enough to set $R = T$ in Theorem (1).

E. Corollary 3.3

Let (X, S, ∗) be a complete S- fuzzy metric space. Let f, R, T: X $\to$X be three self-maps and $\theta: [0, 1) \to (\frac{1}{2}, 1]$ be defined by (2.1). Assume that there exists $r \in [0, 1)$ such that for each$x, y, z \in X$,

$$max\,\{S(x, y, fTx, t), S(x, y, fRx, t)\}^{\theta(r)} \geq S(x, y, z, t)$$

implies

$$\begin{Bmatrix} S(fRx, fRy, fRz, t) * S(fTx, fTy, fTz, t) * \\ S(fRx, fRy, fTz, t) * S(fRz, fTy, fTx, t) \end{Bmatrix}$$

$$\geq S(x, y, z, t)^r$$

And if f is one-to-one, $fR = Rf$ and $fT = Tf$, then f, T and R have a unique common fixed point $v \in X$.

F. Proof

Considering fR and fT as two maps in the given contractive condition of Theorem (3.1), we get a unique common fixed point for fR and fT , i.e., $fRv = fT v = v$. Since f is one-to-one, $fRv = fTv$ implies that $Rv = T v$. From

$$1 = \max\{S(v, v, fTv, t), S(v, v, fRv, t)\}^{\theta(r)} \\ \geq S(v, v, Tv, t)$$

implies

$$\begin{Bmatrix} S(fRv, fRv, fRTv, t) * S(fTv, fTv, fT^2v, t) \\ * S(fRv, fRv, fT^2v, t) * S(fRTv, fTv, fTv, t) \end{Bmatrix}$$

$$= \begin{Bmatrix} S(fRv, fRv, RfTv, t) * S(fTv, fTv, TfTv, t) \\ * S(fRv, fRv, TfTv, t) * S(RfTv, fTv, fTv, t) \end{Bmatrix}$$

$$= \begin{Bmatrix} S(v, v, Rv, t) * S(v, v, Tv, t) * \\ S(v, v, Tv, t) * S(Rv, v, v, t) \end{Bmatrix}$$

$$= S(v, v, Tv, t)$$

$$\geq S(v, v, Tv, t)^r.$$

Therefore $Tv = v$, hence v is common fixed point of f, R and T.

G. Corollary 3.4

Let (X, S, ∗) be a complete S- fuzzy metric space. Let $T, R: X \to X$ be two self maps such that for each $x, y, z \in X$, satisfying the following condition. If there exists $r \in [0, 1)$ such that for each x, y, z ∈ Xsatisfying the condition

$$max\{S(x, y, Tx, t), S(x, y, Rx, t)\}^{\frac{1}{2}} \geq S(x, y, z, t)$$

implies

$$\begin{Bmatrix} S(Rx, Ry, Rz, t) * S(Tx, Ty, Tz, t) \\ * S(Rx, Ry, Tz, t) * S(Tx, Ty, Rz, t) \end{Bmatrix} \geq S(x, y, z, t)^r$$

Then there exists a unique common fixed point of T and R.

H. Proof

It is enough to set $r = \frac{1}{2}$in Theorem 3.1, we get the result.

IV. Conclusion

In this article, we have proved some new type of Suzuki- type fixed point theorem for non-Archimedean S - fuzzy metric spaces which is generalization of Suzuki-Type fixed point results in S – fuzzy metric spaces. Also, we have shown the existence and uniqueness of fixed points for three self maps in the same structure.

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