On an Identity Involving Stirling Numbers of the Second Kind

Kwame Yankson

***Abstract* — We investigate two generalized forms for the recurrence relation $S(n,k) = kS(n-1,k) + S(n-1,k-1)$. From these generalized forms, we derive a new identity, for which proof of the identity is given.**

***Keywords* — Stirling numbers of the Second Kind, recurrence relations, identities, proofs.**

I. Introduction

Stirling Numbers of the Second Kind are a well-known and well-studied aspect of enumerative combinatorics [1]-[4]. They count the number of partitions of a set S of n elements into exactly k non-empty subsets, and can be obtained via the recurrence relation $S(n,k) = kS(n-1,k) + S(n-1,k-1)$. In this paper, we derive an identity [5] for Stirling Numbers of the Second Kind, first by equating two generalizations of $S(n,k)$ to derive the identity, and then proceeding to employ a generating function [6] to prove the identity. In this regard, we follow other works (c.f., [7]-[10]) which have studied combinatorial properties of series that in some cases also include elucidating and proving identities. Given a sequence with successive terms, one way to transform the sequence is to generate a new one where the terms comprise sums of successive differences [3]. In this paper, we recursively apply one equation to get another [3], the equation being the simple form of $S(n,k)$, which then leads to one generalized form of $S(n,k)$. We next conceptualize $S(n,k)$ in terms of set partitions [3], [6] wherein another generalization for $S(n,k)$ thus obtained is proved.

II. Two ways to Conceptualize Stirling Numbers of the Second Kind

One way is that Stirling Numbers of the Second Kind count the number of partitions of a set S of n elements into exactly k non-empty sets. This approach leads to the common recurrence relation for $S(n,k)$ given in the introduction. To generalize that recurrence relation, we can study patterns generated from h iterations of $S(n,k)$. In other words,

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$
$$= k^2S(n-2,k) + (2k-1)S(n-2,k-1) + S(n-2,k-1)$$

After h iterations, we can express $S(n,k)$ in terms of $S\,(n-h,k-1), S\,(n-h,k-2), \ldots\, S\,(n-h,k-i)$ where $0 \le i \le h$. Each term $S\,(n-h\,,k-i\,)$ is multiplied by a polynomial which is the result of applying the rule for the simple recurrence relation for $S(n,k)$ to the $(h-1)$th iteration of $S(n,k)$ and grouping all $S(n-h,k-i)$ terms in the hth iteration.

We are interested in writing the coefficients of the terms in the polynomial multiplying $S(n-h,k-i)$ in the hth iteration of $S(n,k)$, under those of the polynomial multiplying $S(n-h,k-i+1)$ in the $(h-1)$th iteration of $S(n,k)$. This way we create $h+1$ triangles after h iterations of $S(n,k)$ corresponding with the terms $S(n-h,k-1), S(n-h\,,k-2), \ldots, S(n-h\,,k-h)$ expressing $S(n,k)$. The first and last triangles consist of only a column of 1's so in reality there are only $h-1$ triangles formed after h iterations. Below are the triangles that result from 5 iterations of $S(n,k)$.

1 1
1 2 1 1
1 3 3 1 3 3 1
1 4 6 4 1 6 12 7 4 6 1
1 5 10 10 5 1 10 30 35 15 10 30 25 5 10 1

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K. Yankson, Ph.D. and former Postdoc, University of Michigan, USA.
(corresponding e-mail: kyankson@gmail.com)

The interesting observation here is that the triangles shown above are related to Pascal's triangle in that the first column in the first triangle corresponds to the second column in Pascal's triangle. Similarly, the first column in the second triangle corresponds to the third column in Pascal's triangle. Based on these observations, we can deduce that the triangles are formed by truncating Pascal's triangle and combining the result with a corresponding diagonal from the triangle for Stirling Numbers of the Second Kind. For each column of the truncated triangle, we multiply the terms of that column by the number at the top of the column, which depends on the diagonal from the triangle for Stirling Numbers of the Second Kind. These descriptions, when put into equations, lead to a first generalized form for $S(n,k)$.

$$\begin{aligned} S(n,k) &= [\binom{h}{0}S(0,0)k^h - \binom{h}{1}S(1,0)^{h-1} + \cdots \pm \binom{h}{h}S(k,0)k^{h-h}]S(n-h,k) \\ &\quad +[\binom{h}{1}S(1,1)k^{h-1} - \binom{h}{2}S(2,1)^{h-2} + \cdots \pm \binom{h}{h}S(k,1)k^{h-h}]S(n-h,k-1) \\ &\quad + \cdots + \binom{h}{h}S(h,h)S(n-h,k-h) \\ &= \sum_{i=0}^{h} S(n-h,k-i) \sum_{j=i}^{h} \binom{h}{j} S(j,i)k^{h-j}(-1)^{j-i} \end{aligned}$$

Notice that when $h = 1$, the above first generalization reduces to the simple recurrence relation $S(n,k) = kS(n-1,k) + S(n-1,k-1)$.

A second way to conceptualize $S(n,k)$ is that given n elements in a set S, we want to partition S into $h + 1$ sets where S_1, S_2, …, S_h have cardinality 1 and S_{h+1} has cardinality $n — h$, where $n — h > k$. Therefore, for $0 \le h \le k \le n$, we have another generalized form for $S(n,k)$.

$$S(n,k) = \sum_{i=0}^{h} S(n-h,k-i) \sum_{j=0}^{h-i} \binom{h}{j} S(h-j,i)(k-i)^j$$

Let us prove this second generalization:

Proof. Let the set S have n elements $e_1, e_2, \ldots, e_n$. We put $e_1, e_2, \ldots, e_{n-h}$ in S_{h+1} and $e_{n-h+1}, e_{n-h+2}, \ldots, e_n$ each in a set. We then partition S_{h+1} into k sets and insert each of $e_{n-h+1}, e_{n-h+2}, \ldots, e_n$ into any of these sets. There are $k^h S(n — h,k)$ ways to achieve this. This is the term that corresponds with $i = 0$. Note that for $j < h$, $S(h — j,0) = 0$. We can also partition S_{h+1} into $k — 1$ sets, so that for $0 < j < h — 1$ we insert j of $e_{n-h+1}, e_{n-h+2}, \ldots, e_n$ into the $k — 1$ sets. The remaining $e_{n-h+1}, e_{n-h+2}, \ldots, e_n$ are put into the *kth* set. This amounts to:

$$\begin{aligned} &[\binom{h}{0}(k-1)^0 + \binom{h}{1}(k-1)^1 + \cdots + \binom{h}{h-1}(k-1)^{h-1}]S(n-h,k-1) \\ &= S(n-h,k-1) \sum_{j=0}^{h-1} \binom{h}{j} S(h-j,1)(k-1)^j \end{aligned}$$

Or we can partition S_{h+1} into $k — 2$ non-empty subsets each with an element from S_{h+1} and 2 non-empty sets with the remaining elements. Or we can partition S_{h+1} into $k — 3$ subsets applying the same reasoning, and so on. We may go on in this fashion until we keep all $e_{n-h+1}, e_{n-h+2}, \ldots, e_n$ in h sets and partition S_{h+1} into $k — h$ sets. $S(n,k)$ is the number of ways to carry out all these partitions. Hence

$$\begin{aligned} S(n,k) &= S(n-h,k) \sum_{j=0}^{h} \binom{h}{j} S(h-j,0)k^j + S(n-h,k-1) \sum_{j=0}^{h-1} \binom{h}{j} S(h-j,1)(k-1)^j + \cdots \\ &\quad + S(n-h,k-h) \sum_{j=0}^{h-h} \binom{h}{j} S(h-j,h)(k-h)^j \\ &= \sum_{i=0}^{h} S(n-h,k-i) \sum_{j=0}^{h-i} \binom{h}{j} S(h-j,i)(k-i)^j \end{aligned}$$

Notice then that the two generalizations for $S(n,k)$ just given are identical for the first summation but differ for the second. We can thus assume, and then subsequently prove, that although written differently, if the second summations of each generalization are made equal to one another, an interesting identity can result. In other words, for $0 \le h \le k \le n$, we can assume:

$$\sum_{j=i}^{h} (-1)^{j-i}\binom{h}{j}S(j,i)k^{h-j} = \sum_{j=0}^{h-i}\binom{h}{j}S(h-j,i)(k-i)^{j}$$

Starting from the left-hand-side of the above equation, we can expand it to get:

$$\sum_{j=i}^{h}\binom{h}{j}S(j,i)k^{h-j}(-1)^{j-i} = \sum_{t=0}^{h-i}\binom{h}{t}S(h-t,i)\sum_{r=0}^{t}\binom{t}{r}k^{r}(-i)^{t-r}$$

This equation can only be true if the coefficient for k^{h-j} is the same on both sides, so set $r = h - j \rightarrow h - j \le t \le h - i$, in which case we have:

$$\binom{h}{j}S(j,i)(-1)^{j-i} = \sum_{t=h-j}^{h-i}\binom{h}{t}S(h-t,i)\binom{t}{h-j}(-i)^{t-h+j}$$

Further set $h - u = t \rightarrow t = h - u$, so that we have:

$$\binom{h}{j}S(j,i)(-1)^{j-i} = \sum_{u=i}^{j}\binom{h}{u}S(u,i)\binom{h-u}{h-j}(-i)^{j-u}$$

Which then simplifies to:

$$S(j,i) = \sum_{u=i}^{j}\binom{j}{u}S(u,i)i^{j-u}(-1)^{i+u}$$

This resulting identity for Stirling Numbers of the Second Kind is the central subject of this paper. In the next section, we shall prove this identity by drawing on a generating function [6] for Stirling Numbers of the Second Kind in the proof.

III. Proving the Identity

The identity resulting from equating two generalizations of Stirling Numbers of the Second Kind, as shown in the section above, is stated again below:

$$S(j,i) = \sum_{u=i}^{j}\binom{j}{u}S(u,i)i^{j-u}(-1)^{i+u}$$

We shall now prove this identity.
Proof. We seek to show that:

$$\sum_{j=0}^{\infty} S(j,i)\frac{x^j}{j!} = \sum_{j=0}^{\infty}\sum_{u=i}^{j}(-1)^{i+u}\binom{j}{u}S(u,i)i^{j-u}\frac{x^j}{j!}$$

In the equation above, we aim to equate to $\frac{(e^x-1)^i}{i!}$, drawing on a generating function for Stirling Numbers of the Second Kind. Thus, for $i \leq u \leq j < \infty$, we change the bounds for the sums on the right-hand-side:

$$\sum_{u=i}^{\infty} \sum_{j=u}^{\infty} (-1)^{i+u} \binom{j}{u} S(u,i) i^{j-u} \frac{x^j}{j!}$$

It follows that, with further simplification, we get:

$$\sum_{u=i}^{\infty} (-1)^{i+u} S(u,i) i^{-u} \sum_{j=u}^{\infty} \binom{j}{u} \frac{(ix)^j}{j!}$$

Notice however that:

$$\sum_{j=u}^{\infty} \binom{j}{u} \frac{(ix)^j}{j!} = \sum_{j=0}^{\infty} \binom{j+u}{j} \frac{(ix)^{j+u}}{(j+u)!} = \frac{(ix)^u}{u!} e^{ix}$$

So, we can write the right-hand-side of the equation thus as:

$$\sum_{u=i}^{\infty} (-1)^{i+u} S(u,i) i^{-u} \frac{(ix)^u}{u!} e^{ix} = (-1)^i e^{ix} \sum_{u=i}^{\infty} \frac{(-x)^u}{u!} = (-1)^i e^{ix} \frac{(e^{-x}-1)^i}{i!} = \frac{(e^x-1)^i}{i!}$$

IV. Conclusion

In this paper, we have worked to show a new identity derived from equating two generalizations of $S(n,k)$. In combinatorics, a lot of work has already been done on Stirling Numbers [11] as well as in general on studies of the properties of series ([12]-[18]). These include approaches that investigate error bounds [19], special functions [20] as well as specialized series such as q-series [21]. The present work on Stirling Numbers of the Second Kind [22] fits within this overall field of study of combinatorial properties of numbers and of series, for which there are examples of several other interesting works (c.f., [23]-[26]). We hope to contribute in general to the further study of basic combinatorial properties of series, and of recurrence relations such as that of Stirling Numbers of the Second Kind.

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