The Formula of the Trace of Triangle $n \times n$ Matrix to the Power of Positive Integer

Fitri Aryani, Medyantiwi Rahmawita, Megawati, and Sarbaini

***Abstract* — This study determined the general form of the trace of the triangular matrices $n \times n$ with the power of positive integer. Before obtaining the general form of the trace of triangular matrices (upper triangle and lower triangle) $n \times n$ with the power positive integer, first obtain the general form of the triangular matrices $n \times n$ with power positive integer. Obtaining the general form of the triangular matrices $n \times n$ with the power positive integer is carried out by determining of the triangular matrices from power two to power eight. It is further suspected that the general form of a triangular matrices $n \times n$ with the power of a positive integer and prove it using mathematical induction. Finally, a triangular matrices trace $n \times n$ with the power of a positive integer is obtained with direct proof based on the general form of the matrices has been obtained. Given the application trace of the triangle matrices $n \times n$ with power positive integer by an example.**

***Keywords* — Mathematical induction, triangular matrix, matrix power, matrix trace.**

I. INTRODUCTION

Matrix theory is a fundamental theory in algebra widely used in other fields of science, including insurance, economics, biology, chemistry, physics, and others. Many things can be done from a matrix, such as a matrix multiplication, matrix determinants, matrix traces, etc. In this study, what is discussed is to calculate the trace matrix. A trace of the matrix is the sum of the main diagonal elements of a square matrix.

Calculating the trace of a matrix is not so difficult, but if the matrix is powered, then to calculate the trace, its matrix must be multiplied as many times n . Furthermore, the trace of the power matrix can be determined. So calculating the trace of the power matrix is quite complicated. That is, it is pretty interesting to study how to find the right formula for calculating the trace of the power matrix. The power matrix's trace value is obtained without going through a long matrix multiplication process. Enough substituting the matrix entries into the formula.

The calculation of traces of the power matrix has been of much concern. According to [1], traces of the power matrices are often discussed in several areas of mathematics, such as Network Analysis, Number Theory, Dynamic Systems, Matrix Theory, and Differential Equations. The formula of the trace matrix of power was also discussed in 2015 by [2], obtained the trace formula of the matrix of positive integers power. In the paper, there are two forms. First, the matrix trace formula power n even, namely:

$$tr(A^n) = \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)] \cdots [[up\ to\ r\ terms]] \big(det(A)\big)^r \big(tr(A)\big)^{n-2r} \quad (1)$$

Second, the matrix trace formula power n odd, which is:

$$tr(A^n) = \sum_{r=0}^{n-1/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)] \cdots [[up\ to\ r\ terms]] \big(det(A)\big)^r \big(tr(A)\big)^{n-2r} \quad (2)$$

In 2017 [3], it discusses the trace of the 2×2 matrix of the negative integer power. In the paper, there are two forms of trace formula of the power matrix. First, the trace formula of the matrix power for n even, which is:

$$tr(A^n) = \frac{\sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)] \cdots [[up\ to\ r\ terms]] \big(det(A)\big)^r \big(tr(A)\big)^{n-2r}}{\big(det(A)\big)^n} \quad (3)$$

Published on July 28, 2023.
F. Aryani, Department of Mathematics, Universitas Islam Negeri Sultam Syarif Kasim Riau, Indonesia.
(corresponding e-mail: khodijah_fitri@uin-suska.ac.id)
M. Rahmawita, Department of Information Systems, Universitas Islam Negeri Sultam Syarif Kasim Riau, Indonesia.
(e-mail: medyantiwi.rahmawita@uin-suska.ac.id)
Megawati, Department of Information Systems, Universitas Islam Negeri Sultam Syarif Kasim Riau, Indonesia.
(e-mail: megawati@uin-suska.ac.id)
Sarbaini, Department of Mathematics, Universitas Islam Negeri Sultam Syarif Kasim Riau, Indonesia.
(e-mail: sarbaini@uin-suska.ac.id)

Second, the matrix trace formula power n odd, which is:

$$tr(A^n) = \frac{\sum_{r=0}^{n-1/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots[[up\ to\ r\ terms]]\left(det(A)\right)^r \left(tr(A)\right)^{n-2r}}{\left(det(A)\right)^n} \tag{4}$$

Furthermore, the research on calculating the trace of power matrices on particular form matrices of the order 2×2 with of the power negative integer was carried out by [4]. Research continues again for matrix sizes 3×3, 4×4, and 5×5 of the special form symmetric matrices, namely the studies [5], [6], and [7], with of integer power. Still regarding trace of matrices of the power, but a different form of matrix than previously discussed in [8] and [9].

Next the trace of the power matrix, [10] has researched how to get the trace formula of a special matrix 3x3 with of the integer power.The forms of such special matrix are:

$$A_{3x3} = \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix}, \forall\, a.b, c \in \mathbb{R}.$$

The result obtained is the general form of the power matrix and the trace of the matrix with the power of positive integer. For the general form of trace of the matrices, a matrix with the power of a negative integer cannot be determined because the matrix has a determinant equal to zero (det $(A_{3x3}) = 0$), meaning that the matrix A_{3x3} has no inverses.

To prevent the spread of existing problems and to be more directed, a restriction is carried out, namely the matrix studied is the matrix of the upper triangle and the lower triangle, with the following shape:

$$A = \begin{pmatrix} a & a & a & \cdots \\ 0 & a & a & \cdots \\ 0 & 0 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a \end{pmatrix} \text{and } B = \begin{pmatrix} a & 0 & 0 & \cdots \\ a & a & 0 & \cdots \\ a & a & a & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a & a & a & a \end{pmatrix} \tag{5}$$

II. Research Methods

The research method is the steps used by the author in solving the problem in this study, namely to obtain the trace formula of a triangular matrix $n \times n$ with the power of a positive integer. The research methodology for this research is the study of literature. The steps used are as follows:

1. Given the matrices of the upper triangle and the lower triangle at the (5).
2. Determines the upper triangle matrix and the lower triangle matrix from $(A_n)^2$ to $(A_n)^8$.
3. Guesses the general form of the upper triangle and the lower triangle matrix $(A_n)^m$, with m of positive integer.
4. Prove the general form of the $(A_n)^m$, with m of positive integer using mathematical induction.
5. Determines the trace of the $(A_n)^m$, with m of positive integer by using the matrix trace definition.
6. Apply the trace of the $(A_n)^m$, with m of positive integer to the example problem.

The description of the definition of matrix multiplication and power and the theorems relating to the rules of matrix power are in [11], [12], [13], and [14]. The proof of the general form of matrix power uses the rules of mathematical induction, whose elaboration is in [15] and [16]. The elaboration of definitions and theorems related to the rule of matrices power and trace of the matrices is provided in [17], [18], and [19].

.

III. Results and Discussion

The results obtained based on the steps in the method research. There are two results obtained, the first: is the general form of the upper triangular matrix and the lower triangle matrix with the power of positive integer, and the second the traces of the upper triangular and the lower triangle matrix with the power of positive integer.

A. Trace of Upper Triangular Matrix $n \times n$ with of the Power of Positive Integer

The power matrices of the upper triangular matrix as on Equation (5), starts from the $(A_n)^2$ to $(A_n)^8$.

$$(A_n)^2 = \begin{pmatrix} a^2 & 2a^2 & 3a^2 & 4a^2 & \cdots & (n-3)a^2 & (n-2)a^2 & (n-1)a^2 & na^2 \\ 0 & a^2 & 2a^2 & 3a^2 & \cdots & (n-4)a^2 & (n-3)a^2 & (n-2)a^2 & (n-1)a^2 \\ 0 & 0 & a^2 & 2a^2 & \cdots & (n-5)a^2 & (n-4)a^2 & (n-3)a^2 & (n-2)a^2 \\ 0 & 0 & 0 & a^2 & \cdots & (n-6)a^2 & (n-5)a^2 & (n-4)a^2 & (n-3)a^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^2 & 2a^2 & 3a^2 & 4a^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^2 & 2a^2 & 3a^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^2 & 2a^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^2 \end{pmatrix} \tag{6}$$

$$(A_n)^3 = \begin{pmatrix} a^3 & 3a^3 & 6a^3 & 10a^3 & \cdots & \frac{1}{2}(n-3)(n-2)a^3 & \frac{1}{2}(n-2)(n-1)a^3 & \frac{1}{2}(n-1)(n)a^3 & \frac{1}{2}n(n+1)a^3 \\ 0 & a^3 & 3a^3 & 6a^3 & \cdots & \frac{1}{2}(n-4)(n-3)a^3 & \frac{1}{2}(n-3)(n-2)a^3 & \frac{1}{2}(n-2)(n-1)a^3 & \frac{1}{2}(n-1)(n)a^3 \\ 0 & 0 & a^3 & 3a^3 & \cdots & \frac{1}{2}(n-5)(n-4)a^3 & \frac{1}{2}(n-4)(n-3)a^3 & \frac{1}{2}(n-3)(n-2)a^3 & \frac{1}{2}(n-2)(n-1)a^3 \\ 0 & 0 & 0 & a^3 & \cdots & \frac{1}{2}(n-6)(n-5)a^3 & \frac{1}{2}(n-5)(n-4)a^3 & \frac{1}{2}(n-4)(n-3)a^3 & \frac{1}{2}(n-3)(n-2)a^3 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^3 & 3a^3 & 6a^3 & 10a^3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^3 & 3a^3 & 6a^3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^3 & 3a^3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^3 \end{pmatrix} \tag{7}$$

$$(A_n)^4 = \begin{pmatrix} a^4 & 4a^4 & 10a^4 & 20a^4 & \cdots & \frac{1}{6}(n-3)(n-2)(n-1)a^4 & \frac{1}{6}(n-2)(n-1)(n)a^4 & \frac{1}{6}(n-1)(n)(n+1)a^4 & \frac{1}{6}n(n+1)(n+2)a^4 \\ 0 & a^4 & 4a^4 & 10a^4 & \cdots & \frac{1}{6}(n-4)(n-3)(n-2)a^4 & \frac{1}{6}(n-3)(n-2)(n-1)a^4 & \frac{1}{6}(n-2)(n-1)(n)a^4 & \frac{1}{6}(n-1)(n)(n+1)a^4 \\ 0 & 0 & a^4 & 4a^4 & \cdots & \frac{1}{6}(n-5)(n-4)(n-3)a^4 & \frac{1}{6}(n-4)(n-3)(n-2)a^4 & \frac{1}{6}(n-3)(n-2)(n-1)a^4 & \frac{1}{6}(n-2)(n-1)(n)a^4 \\ 0 & 0 & 0 & a^4 & \cdots & \frac{1}{6}(n-6)(n-5)(n-4)a^4 & \frac{1}{6}(n-5)(n-4)(n-3)a^4 & \frac{1}{6}(n-4)(n-3)(n-2)a^4 & \frac{1}{6}(n-3)(n-2)(n-1)a^4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^4 & 4a^4 & 10a^4 & 20a^4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^4 & 4a^4 & 10a^4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^4 & 4a^4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^4 \end{pmatrix} \tag{8}$$

$$(A_n)^5 = (a_{ij}) = \begin{cases} a^5, & \text{for } i = 1,2,3,\ldots,n \text{ and } j = i+0 \\ 5a^5, & \text{for } i = 1,2,3,\ldots,n-1 \text{ and } j = i+1 \\ 15a^5, & \text{for } i = 1,2,3,\ldots,n-2 \text{ and } j = i+2 \\ 35a^5, & \text{for } i = 1,2,3,\ldots,n-3 \text{ and } j = i+3 \\ \vdots & \\ \frac{1}{24}(n-3)(n-2)(n-1)(n)a^5, & \text{for } i = 1,2,3,4 \text{ and } j = i+(n-4) \\ \frac{1}{24}(n-2)(n-1)(n)(n+1)a^5, & \text{for } i = 1,2,3 \text{ and } j = i+(n-3) \\ \frac{1}{24}(n-1)(n)(n+1)(n+2)a^5, & \text{for } i = 1,2 \text{ and } j = i+(n-2) \\ \frac{1}{4!}(n)(n+1)(n+2)(n+3)a^5, & \text{for } i = 1, \text{ and } j = i+(n-1) \\ 0, & \text{for } i > j. \end{cases} \tag{9}$$

$$(A_n)^6 = (a_{ij}) = \begin{cases} a^6, & for \quad i = 1,2,3,\ldots,n \quad and \quad j = i+0 \\ 6a^6, & for \quad i = 1,2,3,\ldots,n-1 \quad and \quad j = i+1 \\ 21a^6, & for \quad i = 1,2,3,\ldots,n-2 \quad and \quad j = i+2 \\ 56a^6, & for \quad i = 1,2,3,\ldots,n-3 \quad and \quad j = i+3 \\ \vdots & \\ \frac{1}{120}(n-3)(n-2)(n-1)(n)(n+1)a^6, & for \quad i = 1,2,3,4 \quad and \quad j = i+(n-4) \\ \frac{1}{120}(n-2)(n-1)(n)(n+1)(n+2)a^6, & for \quad i = 1,2,3 \quad and \quad j = i+(n-3) \\ \frac{1}{120}(n-1)(n)(n+1)(n+2)(n+3)a^6, & for \quad i = 1,2 \quad and \quad j = i+(n-2) \\ \frac{1}{5!}(n)(n+1)(n+2)(n+3)(n+4)a^6, & for \quad i = 1, \quad and \quad j = i+(n-1) \\ 0, & for \quad i > j. \end{cases} \tag{10}$$

$$(A_n)^7 = (a_{ij}) = \begin{cases} a^7, & for \quad i = 1,2,3,\ldots,n \quad and \quad j = i+0 \\ 7a^7, & for \quad i = 1,2,3,\ldots,n-1 \quad and \quad j = i+1 \\ 28a^7, & for \quad i = 1,2,3,\ldots,n-2 \quad and \quad j = i+2 \\ 84a^7, & for \quad i = 1,2,3,\ldots,n-3 \quad and \quad j = i+3 \\ \vdots & \\ \frac{1}{6!}(n-3)(n-2)(n-1)(n)(n+1)(n+2)a^7, & for \quad i = 1,2,3,4 \quad and \quad j = i+(n-4) \\ \frac{1}{6!}(n-2)(n-1)(n)(n+1)(n+2)(n+3)a^7, & for \quad i = 1,2,3 \quad and \quad j = i+(n-3) \\ \frac{1}{6!}(n-1)(n)(n+1)(n+2)(n+3)(n+4)a^7, & for \quad i = 1,2 \quad and \quad j = i+(n-2) \\ \frac{1}{6!}(n)(n+1)(n+2)(n+3)(n+4)(n+5)a^7, & for \quad i = 1, \quad and \quad j = i+(n-1) \\ 0, & for \quad i > j.. \end{cases} \tag{11}$$

$$(A_n)^8 = (a_{ij}) = \begin{cases} a^8, & for \quad i = 1,2,3,\ldots,n \quad and \quad j = i+0 \\ 8a^8, & for \quad i = 1,2,3,\ldots,n-1 \quad and \quad j = i+1 \\ 36a^8, & for \quad i = 1,2,3,\ldots,n-2 \quad and \quad j = i+2 \\ 120a^8, & for \quad i = 1,2,3,\ldots,n-3 \quad and \quad j = i+3 \\ \vdots & \\ \frac{1}{7!}(n-3)(n-2)(n-1)(n)(n+1)(n+2)(n+4)a^8, & for \quad i = 1,2,3,4 \quad and \quad j = i+(n-4) \\ \frac{1}{7!}(n-2)(n-1)(n)(n+1)(n+2)(n+3)(n+4)a^8, & for \quad i = 1,2,3 \quad and \quad j = i+(n-3) \\ \frac{1}{7!}(n-1)(n)(n+1)(n+2)(n+3)(n+4)(n+5)a^8, & for \quad i = 1,2 \quad and \quad j = i+(n-2) \\ \frac{1}{7!}(n)(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)a^8, & for \quad i = 1, \quad and \quad j = i+(n-1) \\ 0, & for \quad i > j.. \end{cases} \tag{12}$$

Once the upper triangular power matrix form is obtained from $(A_n)^2$ to $(A_n)^8$, then it can be assumed that the general form of an upper triangular matrix with of the power of positive integer and then will be prove using mathematical induction, presented in the following theorem.

Theorem 1. Given the upper triangle matrix on Equation (5), then upper triangular matrix with the power of a positive integer $m \geq 2$ is obtained, namely:

$$
(A_n)^m = (a_{ij}) = \begin{cases}
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=1-n}^{(m-n-1)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n \quad and \quad j=i+0 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=2-n}^{(m-n)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-1 \quad and \quad j=i+1 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=3-n}^{(m-n+1)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-2 \quad and \quad j=i+2 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=4-n}^{(m-n+2)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-3 \quad and \quad j=i+3 \\
\quad\vdots & \quad\vdots \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-3)-n}^{(m-n+(n-5))}(n+t)\, a^m, & for \quad i=1,2,3,4 \quad and \quad j=i+(n-4) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-2)-n}^{(m-n+(-4))}(n+t)\, a^m, & for \quad i=1,2,3 \quad and \quad j=i+(n-3) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-1)-n}^{(m-n+(n-3))}(n+t)\, a^m, & for \quad i=1,2 \quad and \quad j=i+(n-2) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=n-n}^{(m-n+(n-2))}(n+t)\, a^m, & for \quad i=1, \quad and \quad j=i+(n-1) \\
0, & for \quad i > j..
\end{cases} \tag{13}
$$

Proof:
The proof of the theorem uses the rules of mathematical induction,
Suppose:

$$
p(m): (A_n)^m = (a_{ij}) = \begin{cases}
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=1-n}^{(m-n-1)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n \quad and \quad j=i+0 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=2-n}^{(m-n)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-1 \quad and \quad j=i+1 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=3-n}^{(m-n+1)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-2 \quad and \quad j=i+2 \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=4-n}^{(m-n+2)}(n+t)\, a^m, & for \quad i=1,2,3,\ldots,n-3 \quad and \quad j=i+3 \\
\quad\vdots & \quad\vdots \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-3)-n}^{(m-n+(n-5))}(n+t)\, a^m, & for \quad i=1,2,3,4 \quad and \quad j=i+(n-4) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-2)-n}^{(m-n+(-4))}(n+t)\, a^m, & for \quad i=1,2,3 \quad and \quad j=i+(n-3) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=(n-1)-n}^{(m-n+(n-3))}(n+t)\, a^m, & for \quad i=1,2 \quad and \quad j=i+(n-2) \\
\dfrac{1}{(m-1)!}\displaystyle\prod_{t=n-n}^{(m-n+(n-2))}(n+t)\, a^m, & for \quad i=1, \quad and \quad j=i+(n-1) \\
0, & for \quad i > j..
\end{cases} \tag{14}
$$

It will be indicated for $p(2)$ true, namely:

$$p(2):\left(A_n\right)^2=\left(a_{ij}\right)=\begin{cases}\frac{1}{(2-1)!}\prod_{t=1-n}^{(1-n)}(n+t)\,a^2, & for\quad i=1,2,3,\dots,n \quad and\quad j=i+0\\ \frac{1}{(2-1)!}\prod_{t=2-n}^{(2-n)}(n+t)\,a^2, & for\quad i=1,2,3,\dots,n-1 \quad and\quad j=i+1\\ \frac{1}{(2-1)!}\prod_{t=3-n}^{(3-n)}(n+t)\,a^2, & for\quad i=1,2,3,\dots,n-2 \quad and\quad j=i+2\\ \frac{1}{(2-1)!}\prod_{t=4-n}^{(4-n)}(n+t)\,a^2, & for\quad i=1,2,3,\dots,n-3 \quad and\quad j=i+3\\ \quad\vdots & \quad\quad\vdots\\ \frac{1}{(2-1)!}\prod_{t=(n-3)-n}^{-3}(n+t)\,a^2, & for\quad i=1,2,3,4 \quad and\quad j=i+(n-4)\\ \frac{1}{(2-1)!}\prod_{t=(n-2)-n}^{-2}(n+t)\,a^2, & for\quad i=1,2,3 \quad and\quad j=i+(n-3)\\ \frac{1}{(2-1)!}\prod_{t=(n-1)-n}^{-1}(n+t)\,a^2, & for\quad i=1,2 \quad and\quad j=i+(n-2)\\ \frac{1}{(2-1)!}\prod_{t=n-n}^{0}(n+t)\,a^2, & for\quad i=1, \quad and\quad j=i+(n-1)\\ 0, & for\quad i>j.\end{cases}$$

(15)

Or it can be written as:

$$p(2):\left(A_n\right)^2=\left(a_{ij}\right)=\begin{cases}a^2, & for\quad i=1,2,3,\dots,n \quad and\quad j=i+0\\ 2a^2, & for\quad i=1,2,3,\dots,n-1 \quad and\quad j=i+1\\ 3a^2, & for\quad i=1,2,3,\dots,n-2 \quad and\quad j=i+2\\ 4a^2, & for\quad i=1,2,3,\dots,n-3 \quad and\quad j=i+3\\ \quad\vdots & \quad\quad\vdots\\ (n-3)a^2, & for\quad i=1,2,3,4 \quad and\quad j=i+(n-4)\\ (n-2)a^2, & for\quad i=1,2,3 \quad and\quad j=i+(n-3)\\ (n-1)a^2, & for\quad i=1,2 \quad and\quad j=i+(n-2)\\ na^2, & for\quad i=1, \quad and\quad j=i+(n-1)\\ 0, & for\quad i>j..\end{cases} \tag{16}$$

If you pay attention to the form above, it is the same as the matrix form in the power of two in Equation (6), namely:

$$\left(A_n\right)^2=\begin{pmatrix} a^2 & 2a^2 & 3a^2 & 4a^2 & \cdots & (n-3)a^2 & (n-2)a^2 & (n-1)a^2 & na^2\\ 0 & a^2 & 2a^2 & 3a^2 & \cdots & (n-4)a^2 & (n-3)a^2 & (n-2)a^2 & (n-1)a^2\\ 0 & 0 & a^2 & 2a^2 & \cdots & (n-5)a^2 & (n-4)a^2 & (n-3)a^2 & (n-2)a^2\\ 0 & 0 & 0 & a^2 & \cdots & (n-6)a^2 & (n-5)a^2 & (n-4)a^2 & (n-3)a^2\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & a^2 & 2a^2 & 3a^2 & 4a^2\\ 0 & 0 & 0 & 0 & \cdots & 0 & a^2 & 2a^2 & 3a^2\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^2 & 2a^2\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^2 \end{pmatrix} \tag{17}$$

Then $p(2)$ right.

(i) Assume for $p(k)$ with $k \geq 2$ true, i.e.:

$$p(k):\left(A_n\right)^k=\left(a_{ij}\right)=\begin{cases} \dfrac{1}{(k-1)!}\displaystyle\prod_{t=1-n}^{(k-n-1)}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3,\ldots,n \quad \textit{and} \quad j=i+0 \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=2-n}^{(k-n)}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3,\ldots,n-1 \quad \textit{and} \quad j=i+1 \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=3-n}^{(k-n+1)}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3,\ldots,n-2 \quad \textit{and} \quad j=i+2 \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=4-n}^{(k-n+2)}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3,\ldots,n-3 \quad \textit{and} \quad j=i+3 \\ \quad\vdots & \qquad\vdots \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=(n-3)-n}^{(k-n+(n-5))}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3,4 \quad \textit{and} \quad j=i+\left(n-4\right) \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=(n-2)-n}^{(k-n+(-4))}\left(n+t\right)a^k, & \textit{for} \quad i=1,2,3 \quad \textit{and} \quad j=i+\left(n-3\right) \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=(n-1)-n}^{(k-n+(n-3))}\left(n+t\right)a \ , & \textit{for} \quad i=1,2 \quad \textit{and} \quad j=i+\left(n-2\right) \\ \dfrac{1}{(k-1)!}\displaystyle\prod_{t=n-n}^{(k-n+(n-2))}\left(n+t\right)a^k, & \textit{for} \quad i=1, \quad \textit{and} \quad j=i+\left(n-1\right) \\ 0, & \textit{for} \quad i>j.. \end{cases} \tag{18}$$

And will be proved for $p(k+1)$ also true, i.e.:

$$p(k+1):\left(A_n\right)^{k+1}=\left(a_{ij}\right)=\begin{cases} \dfrac{1}{(k)!}\displaystyle\prod_{t=1-n}^{(k-n)}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3,\ldots,n \quad \textit{and} \quad j=i+0 \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=2-n}^{(k-n+1)}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3,\ldots,n-1 \quad \textit{and} \quad j=i+1 \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=3-n}^{(k-n+2)}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3,\ldots,n-2 \quad \textit{and} \quad j=i+2 \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=4-n}^{(k-n+3)}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3,\ldots,n-3 \quad \textit{and} \quad j=i+3 \\ \quad\vdots & \qquad\vdots \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=(n-3)-n}^{(k-n+(n-4))}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3,4 \quad \textit{and} \quad j=i+\left(n-4\right) \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=(n-2)-n}^{(k-n+(-3))}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2,3 \quad \textit{and} \quad j=i+\left(n-3\right) \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=(n-1)-n}^{(k-n+(n-2))}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1,2 \quad \textit{and} \quad j=i+\left(n-2\right) \\ \dfrac{1}{(k)!}\displaystyle\prod_{t=n-n}^{(k-n+(n-1))}\left(n+t\right)a^{k+1}, & \textit{for} \quad i=1, \quad \textit{and} \quad j=i+\left(n-1\right) \\ 0, & \textit{for} \quad i>j.. \end{cases} \tag{19}$$

The proof start from:

$$\left(A_n\right)^{k+1}=\left(A_n\right)^k\left(A_n\right) \tag{20}$$

$$\left(A_n\right)^{k+1}=\left(\begin{pmatrix} a & a & a & a & \cdots & a & a & a & a \\ 0 & a & a & a & \cdots & a & a & a & a \\ 0 & 0 & a & a & \cdots & a & a & a & a \\ 0 & 0 & 0 & a & \cdots & a & a & a & a \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix}\right)^k \begin{pmatrix} a & a & a & a & \cdots & a & a & a & a \\ 0 & a & a & a & \cdots & a & a & a & a \\ 0 & 0 & a & a & \cdots & a & a & a & a \\ 0 & 0 & 0 & a & \cdots & a & a & a & a \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix} \tag{21}$$

a) for main diagonal entries $i = 1,2,3, \dots n$ and $j = i + 0$.

Based on the row entries on $\left(A_n\right)^k$ and the column entries on $\left(A_n\right)$, then the multiplication result for the main diagonal entries are $\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n)}(n+t)\, a^{k+1}$. Because there is a multiplication of 0 as much $n-1$ terms and $\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n-1)}(n+t)\, a^{k}.a$ as much one term, then the result is: $\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n)}(n+t)\, a^{k+1}$.

b) for entries $i = 1,2,3, \dots n-1$ and $j = i + 1$.

Based on the row entries on $\left(A_n\right)^k$ and the column entries on $\left(A_n\right)$, then the multiplication result for those entries are $\frac{1}{(k)!}\prod_{t=2-n}^{(k-n+1)}(n+t)\, a^{k+1}$. Because there is a multiplication of 0 as much $n-2$ terms and the remaining 2 terms are $\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n-1)}(n+t)\, a^{k}.a + \frac{1}{(k-1)!}\prod_{t=2-n}^{(k-n)}(n+t)\, a^{k}.a$, then the result is: $\frac{1}{(k)!}\prod_{t=2-n}^{(k-n+1)}(n+t)\, a^{k+1}$.

c) for entries $i = 1,2,3, \dots n-2 \textit{ and } j = i + 2$

Based on the row entries on $\left(A_n\right)^k$ and the column entries on $\left(A_n\right)$, then the multiplication result for those entries are that there is a multiplication of 0 as much $n-3$ terms and the remaining 3 terms are:

$$\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n-1)}(n+t)\, a^{k}.a + \frac{1}{(k-1)!}\prod_{t=2-n}^{(k-n)}(n+t)\, a^{k}.a + \frac{1}{(k-1)!}\prod_{t=3-n}^{(k-n+1)}(n+t)\, a^{k}.a\ ,$$

then the result is: $\frac{1}{(k)!}\prod_{t=3-n}^{(k-n+2)}(n+t)\, a^{k+1}$.

d) for entries $i = 1,2,3, \dots n-3$ dan $j = i + 3$

Based on the row entries $\left(A_n\right)^k$ and the column entries $\left(A_n\right)$, the multiplication result for those entries are that there is a multiplication of 0 as much $n-4$ terms, and the remaining 4 terms are:

$$\frac{1}{\left(k-1\right)!}\prod_{t=1-n}^{\left(k-n-1\right)}\left(n+t\right)a^{k}.a+\frac{1}{\left(k-1\right)!}\prod_{t=2-n}^{\left(k-n\right)}\left(n+t\right)a^{k}.a+\frac{1}{\left(k-1\right)!}\prod_{t=3-n}^{\left(k-n+1\right)}\left(n+t\right)a^{k}.a+\frac{1}{\left(k-1\right)!}\prod_{t=4-n}^{\left(k-n+2\right)}\left(n+t\right)a^{k}.a \tag{22}$$

then the result is:

$$\frac{1}{\left(k\right)!}\prod_{t=4-n}^{\left(k-n+3\right)}\left(n+t\right)a^{k+1} \tag{23}$$

e) If you look at it from a) to d), there appears to be a pattern that can be determined for subsequent entries. Shown for the last two entries.

f) You for entries $i = 1,2$ dan $j = i + (n-2)$

Based on the row entries on $\left(A_n\right)^k$ and the column entries on $\left(A_n\right)$, then the multiplication result for those entries are that there a multiplication by 0 as much 1 term, and the remaining terms $n-1$ are:

$$\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n-1)}(n+t)a^{k}.a+\frac{1}{(k-1)!}\prod_{t=2-n}^{(k-n)}(n+t)a^{k}.a+\frac{1}{(k-1)!}\prod_{t=3-n}^{(k-n+1)}(n+t)a^{k}.a$$
$$+\frac{1}{(k-1)!}\prod_{t=4-n}^{(k-n+2)}(n+t)a^{k}.a+..+\frac{1}{(k-1)!}\prod_{t=(n-3)-n}^{(k-n+(n-5))}(n+t)a^{k}a+\frac{1}{(k-1)!}\prod_{t=(n-2)-n}^{(k-n+(n-4))}(n+t)a^{k}a \quad (24)$$
$$+.\frac{1}{(k-1)!}\prod_{t=(n-1)-n}^{(k-n+(n-3))}(n+t)a^{k}a$$

then the result is:

$$\frac{1}{(k)!}\prod_{t=(n-1)-n}^{(k-n+(n-2))}(n+t)a^{k+1} \quad (25)$$

g) for entries $i = 1$ dan $j = i + (n-1)$

Based on the row entries on $(A_n)^k$ and the column entries on (A_n), then the multiplication result for those entries are that there not multiplication by 0, and the entries in the first row on $(A_n)^k$ multiplied by the entries in the column are (A_n).

$$\frac{1}{(k-1)!}\prod_{t=1-n}^{(k-n-1)}(n+t)a^{k}.a+\frac{1}{(k-1)!}\prod_{t=2-n}^{(k-n)}(n+t)a^{k}.a+\frac{1}{(k-1)!}\prod_{t=3-n}^{(k-n+1)}(n+t)a^{k}.a$$
$$+\frac{1}{(k-1)!}\prod_{t=4-n}^{(k-n+2)}(n+t)a^{k}.a+..+\frac{1}{(k-1)!}\prod_{t=(n-3)-n}^{(k-n+(n-5))}(n+t)a^{k}a$$
$$+\frac{1}{(k-1)!}\prod_{t=(n-2)-n}^{(k-n+(n-4))}(n+t)a^{k}a+\frac{1}{(k-1)!}\prod_{t=(n-1)-n}^{(k-n+(n-3))}(n+t)a^{k}a \quad (26)$$
$$+\frac{1}{(k-1)!}\prod_{t=n-n}^{(k-n+(n-2))}(n+t)a^{k}a$$

then the result is:

$$\frac{1}{(k)!}\prod_{t=n-n}^{(k-n+(n-1))}(n+t)a^{k+1} \quad (27)$$

such matrix entries can be presented in the form of:

$$p(k+1):(A_n)^{k+1}=(a_{ij})=\begin{cases}\frac{1}{(k)!}\prod_{t=1-n}^{(k-n)}(n+t)a^{k+1}, & for\ \ i=1,2,3,...,n \quad and\ \ j=i+0\\ \frac{1}{(k)!}\prod_{t=2-n}^{(k-n+1)}(n+t)a^{k+1}, & for\ \ i=1,2,3,...,n-1 \quad and\ \ j=i+1\\ \frac{1}{(k)!}\prod_{t=3-n}^{(k-n+2)}(n+t)a^{k+1}, & for\ \ i=1,2,3,...,n-2 \quad and\ \ j=i+2\\ \frac{1}{(k)!}\prod_{t=4-n}^{(k-n+3)}(n+t)a^{k+1}, & for\ \ i=1,2,3,...,n-3 \quad and\ \ j=i+3\\ \quad\vdots & \quad\vdots\\ \frac{1}{(k)!}\prod_{t=(n-3)-n}^{(k-n+(n-4))}(n+t)a^{k+1}, & for\ \ i=1,2,3,4 \quad and\ \ j=i+(n-4)\\ \frac{1}{(k)!}\prod_{t=(n-2)-n}^{(k-n+(-3))}(n+t)a^{k+1}, & for\ \ i=1,2,3 \quad and\ \ j=i+(n-3)\\ \frac{1}{(k)!}\prod_{t=(n-1)-n}^{(k-n+(n-2))}(n+t)a^{k+1}, & for\ \ i=1,2 \quad and\ \ j=i+(n-2)\\ \frac{1}{(k)!}\prod_{t=n-n}^{(k-n+(n-1))}(n+t)a^{k+1}, & for\ \ i=1, \quad and\ \ j=i+(n-1)\\ 0, & for\ \ i>j..\end{cases} \quad (28)$$

The proof is complete.

Furthermore, can be obtained the trace of the upper triangular matrix with the power in the positive integer is expressed on Remark 1 as follows.

Remark 1 Given the upper triangle matrix in Equation (5) i.e.:

$$\left(A_n\right)=\begin{pmatrix} a & a & a & a & \cdots & a & a & a & a \\ 0 & a & a & a & \cdots & a & a & a & a \\ 0 & 0 & a & a & \cdots & a & a & a & a \\ 0 & 0 & 0 & a & \cdots & a & a & a & a \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix} \tag{29}$$

then the trace of the upper triangular matrix with the power of positive integer is obtained:

$$tr\left(A_n\right)^m = n.a^m \text{ with } m \geq 2. \tag{30}$$

Proof:

The proof of the Theorem uses direct proof.

Based on Theorem 1 the main diagonal entries of the upper triangular matrix with the power of positive integer is $\frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)a^m$ from the entry a_{11} until a_{nn}. This means that obtaining traces from the matrix is the same as summing each entry on its main diagonal, namely:

$$\begin{aligned} tr\left(A_n\right)^m &= \underbrace{\frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)a^m + \frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)a^m + \ldots + \frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)a^m}_{n} \\ &= n.\frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)a^m \\ &= n.a^m \end{aligned} \tag{31}$$

The proof is complete.

B. Trace of Lower Triangular Matrix with of the Power of Positive Integer

Similarly, to section A, based on the lower triangular matrix as in (5) and then determined form from $\left(B_n\right)^2$ to $\left(B_n\right)^8$. Furthermore, with looking at the recursive pattern from $\left(B_n\right)^2$ to $\left(B_n\right)^8$, then general form of the lower triangular matrix with the power of positive integer can be guessed. After obtaining the estimation of the general form, then the further proof is carried out, which is stated in the following Theorem.

Theorem 2 Given the lower triangle matrix on Equation (5), then upper triangular matrix with the power of a positive integer $m \geq 2$ is obtained, namely:

$$
(A_n)^m = (a_{ij}) = \begin{cases}
\frac{1}{(m-1)!}\prod_{t=1-n}^{(m-n-1)}(n+t)\,a^m, & for\quad j=1,2,3,\dots,n \quad and \quad i=j+0 \\
\frac{1}{(m-1)!}\prod_{t=2-n}^{(m-n)}(n+t)\,a^m, & for\quad j=1,2,3,\dots,n-1 \quad and \quad i=j+1 \\
\frac{1}{(m-1)!}\prod_{t=3-n}^{(m-n+1)}(n+t)\,a^m, & for\quad j=1,2,3,\dots,n-2 \quad and \quad i=j+2 \\
\frac{1}{(m-1)!}\prod_{t=4-n}^{(m-n+2)}(n+t)\,a^m, & for\quad j=1,2,3,\dots,n-3 \quad and \quad i=j+3 \\
\vdots & \vdots \\
\frac{1}{(m-1)!}\prod_{t=(n-3)-n}^{(m-n+(n-5))}(n+t)\,a^m, & for\quad j=1,2,3,4 \quad and \quad i=j+(n-4) \\
\frac{1}{(m-1)!}\prod_{t=(n-2)-n}^{(m-n+(-4))}(n+t)\,a^m, & for\quad j=1,2,3 \quad and \quad i=j+(n-3) \\
\frac{1}{(m-1)!}\prod_{t=(n-1)-n}^{(m-n+(n-3))}(n+t)\,a^m, & for\quad j=1,2 \quad and \quad i=j+(n-2) \\
\frac{1}{(m-1)!}\prod_{t=n-n}^{(m-n+(n-2))}(n+t)\,a^m, & for\quad j=1, \quad and \quad i=j+(n-1) \\
0, & for\quad j > i
\end{cases} \tag{32}
$$

Proof:

The proof of the Theorem uses the rules of mathematical induction, which are treated the same as those of Theorem 1. The difference is only of the entries location of the matrix form. The proof is complete.

After obtaining the lower triangular matrix form with of the power positive integer at Theorem 2, then the trace of the matrix can be obtained explained in remark 2.

Result 2 Given the lower triangle matrix in Equation (5) i.e.:

$$
(B_n) = \begin{pmatrix}
a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a & a & a & 0 & \cdots & 0 & 0 & 0 & 0 \\
a & a & a & a & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
a & a & a & a & \cdots & a & 0 & 0 & 0 \\
a & a & a & a & \cdots & a & a & 0 & 0 \\
a & a & a & a & \cdots & a & a & a & 0 \\
a & a & a & a & \cdots & a & a & a & a
\end{pmatrix} \tag{33}
$$

then the trace of the lower triangular matrix with the power of positive integer is obtained:

$$
tr\,(B_n)^m = n.a^m \text{ with } m \geq 2. \tag{34}
$$

Proof:

The proof of the theorem uses direct proof.

Based on Theorem 2 the main diagonal entries of the lower triangular matrix with the power of positive integer is a^m from the entry b_{11} until b_{nn}. This means that obtaining b_{nn} traces from the matrix is the equal to summing each entry on its main diagonal. This means to add up a^m as much n, that is:

$$
tr\,(B_n)^m = \underbrace{a^m + a^m + a^m + \dots + a^m}_{n} = n.a^m \tag{35}
$$

The proof is complete.

C. *Application of Trace of Triangle Matrices with of the Power of Positive Integer*

Based on the discussion above, the following will be given some examples of problems that use Theorem 1 to Theorem 2 and Remark 1 to Remark 2.

Example 1. Given the upper triangular matrix as follows:

$$A_7 = \begin{pmatrix} -3 & -3 & -3 & -3 & -3 & -3 & -3 \\ 0 & -3 & -3 & -3 & -3 & -3 & -3 \\ 0 & 0 & -3 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{36}$$

Then specify $(A_7)^{11}$ and $tr\left(A_7^{11}\right)$!

Solution: Using Theorem 1, the matrix power form is obtained:

$$(A_7)^{11} = (a_{ij}) = \begin{cases} (-3)^{11}, & for \quad i = 1,2,3,\dots,7 \quad and \quad j = i+0 \\ 11(-3)^{11}, & for \quad i = 1,2,3,\dots,6 \quad and \quad j = i+1 \\ 66(-3)^{11}, & for \quad i = 1,2,3,\dots,5 \quad and \quad j = i+2 \\ 286(-3)^{11}, & for \quad i = 1,2,3,4 \quad and \quad j = i+3 \\ 1001(-3)^{11}, & for \quad i = 1,2,3 \quad and \quad j = i+4 \\ 3003(-3)^{11}, & for \quad i = 1,2 \quad and \quad j = i+5 \\ 8008(-3)^{11}, & for \quad i = 1 \quad and \quad j = i+6 \\ 0, & i > j. \end{cases} \tag{37}$$

And based on Remark 1, it $tr\left(A_7^{11}\right)$ is:

$$tr\left(A_7^{11}\right) = 7.(-3)^{11} \tag{38}$$

Example 2. Given the lower triangular matrix as follows:

$$B_9 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{39}$$

Then specify $(B_9)^{19}$ and $tr\left(B_9^{19}\right)$!

Solution: Using Theorem 2, the matrix power form is obtained:

$$(B_9)^{19} = (b_{ij}) = \begin{cases} \left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,...,9 \quad and \quad i = j+0 \\ 19\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,...,8 \quad and \quad i = j+1 \\ 190\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,...,7 \quad and \quad i = j+2 \\ 1330\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,...,6 \quad and \quad i = j+3 \\ 7315\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,...,5 \quad and \quad i = j+4 \\ 33649\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3,4 \quad and \quad i = j+5 \\ 134596\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2,3 \quad and \quad i = j+6 \\ 480700\left(\frac{1}{2}\right)^{19}, & for \quad j = 1,2 \quad and \quad i = j+7 \\ 1562275\left(\frac{1}{2}\right)^{19}, & for \quad j = 1, \quad and \quad i = j+8 \\ 0, & for \quad j > i. \end{cases} \tag{40}$$

And based on Remark 2, it $tr\left(B_9^{19}\right)$ is: $tr\left(B_9^{19}\right) = 9.\left(\frac{1}{2}\right)^{19}$

IV. CONCLUSION

According to the results and discussion above, it can be concluded that general formula of the upper triangular matrix and the lower triangle with of the power of positive integers is the same, but only the location of the entries is different. Meanwhile, the general formula for the upper triangular and lower triangular trace matrices with positive integer powers has the same trace value, namely:

$$tr\left(A_n\right)^m = tr\left(B_n\right)^m = n.a^m \tag{41}$$

REFERENCES

[1] Brezinski C, Fika P, Mitrouli M. Estimations of the Trace of Powers of Positive Self-Adjoint Operators by Extrapolation of the Moments. *Electronic Transactions on Numerical Analysis*. 2012 May 7; 39:144–155.
[2] Pahade J, Jha M. Trace of Positive Integer Power of Real 2 × 2 Matrices. *Advances in Linear Algebra & Matrix Theory*. 2015 Desember; 5 (4): 150–155.
[3] Aryani F, Solihin M. Trace Matriks Real Berpangkat Bilangan Bula Negatif, *Jurnal Sains Matematika dan Statistika.* 2017 Juli; 3 (2): 16–23.
[4] Aryani F, Yulianis. Trace Matriks Berbentuk Khusus 2×2 Berpangkat Bilangan Bulat Negatif. *Jurnal Sains Matematika dan Statistika*. 2018 Juli; 4 (2): 105–113.
[5] Aryani F, Cenia PB, Muda Y, Zukrianto Trace Matriks Simetris Berbentuk Khusus Orde 3 Berpangkat Bilangan Bulat. *Prosiding Nasional pada Seminar Nasional Teknologi Informasi, Komunikasi dan Industri* (*SNTIKI*); 2021 Nov 18; (13): 300–310.
[6] Aryani F, Harnita, Muda Y, Zukrianto. Trace Matriks Simetris Berbentuk Khusus 4 x 4 Berpangkat Bilangan Bulat. *Prosiding Nasional pada Seminar Nasional Teknologi Informasi, Komunikasi dan Industri* (*SNTIKI*); 2021 Nov 18; (13): 311–321.
[7] Aryani F, Alfinov SP, Marzuki CC, Rahma AN, Trace Matriks Simetris Berbentuk Khusus 5 x 5 Berpangkat Bilangan Bulat. (*SNTIKI*); 2021 Nov18; (13): 322–333.
[8] Rahmawati, Putri NA, Aryani F, Rahma AN. Trace Matriks Toeplitz Simetris Bentuk Khusus 3×3 Berpangkat Bilangan Bulat Positif. *Jurnal Sains Matematika dan Statistika*. 2019 Juli; 5 (2): 61–70.
[9] Aryani F, Andesta R, Marzuki CC. Trace Matriks Berbentuk Khusus 3×3 Berpangkat Bilangan Bulat Positif. *Jurnal Sains Matematika dan Statistika*. 2020 Januari; 6 (1): 40–9.
[10] Aryani F, Taslim R. Trace Matrix 3 x 3 Berpangkat Bilangan Bulat. *Jurnal Sains Matematika dan Statistika*. 2021 Januari; 7 (1): 1–9.
[17] Marjono. *Linear Algebra*. Malang UB Press; 2012.
[18] Gentle JE. *Matrix algebra*, vol. 10. Springer; 2007.
[19] Kariadinata R. *Algebra of Elementary Matrices*. Bandung Pustaka Setia; 2013.
[20] Larson R. *Elementary Linear Algebra*. 7th ed. Boston Cengage Learning; 2013.

[21] Rosen KH. *Discrete Mathematics and Its Applications*. New York Mc Graw Hill; 2007.
[22] Munir R. *Discrete Mathematics*. Bandung Informatics ITB; 2005.
[23] Rifa'I R. Basic *Matrix Algebra*. Yogyakarta Budi Utama; 2016.
[24] Banerjee S, Roy A. *Linear Algebra and Matrix Analysis for Statistics*. Crc Press Boca Raton; 2014.
[25] Anton H, Rorres C. *Elementary Linear Algebra*. United States of America Wiley; 2013.