The Lie Group Structure of Genus Two Hyperelliptic $\wp$ Functions

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Abstract

We consider the generalized dual transformation for hyperelliptic $\wp$ functions. For the genus two case, by constructing a quadratic invariant form, we find that hyperelliptic $\wp$ functions have the SO(3,2) $\cong$ Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie group structure.

Index Terms

Generalized dual transformation, Genus two hyperelliptic $\wp$ function, SO(3,2) $\cong$ Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie group structure, Higher dimensional KdV equation.

I. Introduction

Some special type of non-linear differential equations can be solved exactly and further provide a series of infinitely many solutions. We are interested in those "solvable/integrable" mechanisms.

Soliton equations are examples of such equations, hence various methods for studying soliton systems are beneficial for our objective. Starting from the inverse scattering method [1]- [3], the soliton theory has many interesting developments, such as the AKNS formulation [4], geometrical approach [5]- [7], Bäcklund transformation [8]- [10], Hirota equation [11], [12], Sato theory [13], vertex construction of the soliton solution [14]- [16], and Schwarzian type mKdV/KdV equation [17].

Non-linear integrable models imply the existence of the potential. KdV equation which is a typical soliton equation has a solution of the Weierstrass $\wp$ function. The τ function is considered as a potential of the KdV equation in the form $u(x-vt) = -2\partial_x^2 \log \tau(x-vt)$, and the τ function corresponds to the σ function of the Weierstrass $\wp$ function, $\wp(u) = -\partial_u^2 \log \sigma(u)$. Thus, differential equations of hyperelliptic $\wp$ functions are the natural generalization of higher dimensional non-linear integrable models, and the σ function plays a role of potential. Hence, hyperelliptic $\wp$ functions are expected to have an optimal property to examine Lie group structures.

We expect that there is a Lie group structure behind some non-linear differential equations, which may be a reason why such non-linear differential equations have infinitely many solutions. Here an addition formula of the Lie group structure might be essential. As the representation of the addition formula of the Lie group, algebraic functions such as trigonometric/elliptic/hyperelliptic functions will emerge for solutions of special differential equations.

The AKNS formalism for the Lax pair is a powerful tool to examine the Lie algebra structure of soliton equations in non-linear integrable models. In our previous researches, we deduced the SO(2,1) $\cong$ Sp(2,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie algebra structure for two-dimensional KdV/ mKdV/ sinh-Gordon models [18]- [23]. Owing to the fact that the KdV equation has the solution of the elliptic $\wp$ function, we deduced that the genus one elliptic $\wp$ function had the SO(2,1) $\cong$ Sp(2,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie algebra structure. In addition, observing the SO(3,2) $\cong$ Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie algebra structure for the two-flows (two-dimensional) Kowalevski top [24], we found that genus two hyperelliptic $\wp$ functions possessed the SO(3,2) $\cong$ Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie algebra structure. By directly using the algebraic addition formula of genus two $\wp$ functions, we obtained the degree two Sp(4,$\mathbf{R}$) Lie group structure [25].

For the general hyperelliptic differential equations, the Lax pair, especially the AKNS formalism, is not known. Thus we directly study the algebraic addition formula and differential equations themselves to find Lie group structure behind. In this study, we use the generalized dual transformation (GDT) to study Lie group structures of genus two hyperelliptic $\wp$ functions.

II. The Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ $\cong$ SO(3,2) Lie Group Structure of Genus Two Hyperelliptic $\wp_{ij}$ Functions

We parametrize the genus two hyperelliptic curve on $\mathbf{R}$ in the form:

$$y^2 = \sum_{n=0}^{6} \lambda_n x^n = \sum_{n=0}^{6} {}_6C_n a_n x^n = a_6 x^6 + 6a_5 x^5 + 15a_4 x^4 + 20a_3 x^3 + 15a_2 x^2 + 6a_1 x + a_0, \tag{1}$$

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where we put $a_6 = 0$ in the end. The Jacobi's inversion problem is the problem to express the symmetric combination of x_1 and x_2 as the function of u_1 and u_2 by using relations:

$$\mathrm{d}u_1 = \frac{\mathrm{d}x_1}{y_1} + \frac{\mathrm{d}x_2}{y_2}, \quad \mathrm{d}u_2 = \frac{x_1\mathrm{d}x_1}{y_1} + \frac{x_2\mathrm{d}x_2}{y_2}. \tag{2}$$

From above relations, we obtain:

$$\frac{\partial x_1}{\partial u_2} = \frac{y_1}{x_1 - x_2}, \quad \frac{\partial x_2}{\partial u_2} = -\frac{y_2}{x_1 - x_2}, \quad \frac{\partial x_1}{\partial u_1} = -\frac{x_2 y_1}{x_1 - x_2}, \quad \frac{\partial x_2}{\partial u_1} = \frac{x_1 y_2}{x_1 - x_2}. \tag{3}$$

Thus, we obtain:

$$\frac{\partial(x_1 + x_2)}{\partial u_1} = -\frac{\partial(x_1 x_2)}{\partial u_2}. \tag{4}$$

As the solution of the Jacobi's inversion problem, we define:

$$\wp_{22}(u_1, u_2) = \frac{\lambda_5}{4}(x_1 + x_2), \quad \wp_{21}(u_1, u_2) = -\frac{\lambda_5}{4} x_1 x_2, \tag{5}$$

and Eq.(4) provides the integrability condition of genus two hyperelliptic $\wp$ functions:

$$\frac{\partial \wp_{22}(u_1, u_2)}{\partial u_1} = \frac{\partial \wp_{21}(u_1, u_2)}{\partial u_2}.$$

Furthermore, if we define [26], [27]:

$$\begin{aligned} \wp_{11}(u_1.u_2) = &\quad \frac{F(x_1, x_2) - 2y_1 y_2}{4(x_1 - x_2)^2}, \\ F(x_1, x_2) = &\quad 2a_6 x_1^3 x_2^3 + 6a_5 x_1^2 x_2^2 (x_1 + x_2) + 30a_4 x_1^2 x_2^2 + 20a_3 x_1 x_2 (x_1 + x_2) + 30a_2 x_1 x_2 \\ &\quad + 6a_1(x_1 + x_2) + 2a_0, \end{aligned} \tag{6}$$

we obtain full integrability conditions:

$$\frac{\partial \wp_{22}(u_1, u_2)}{\partial u_1} = \frac{\partial \wp_{21}(u_1, u_2)}{\partial u_2}, \quad \frac{\partial \wp_{21}(u_1, u_2)}{\partial u_1} = \frac{\partial \wp_{11}(u_1, u_2)}{\partial u_2}. \tag{7}$$

Next, we define another genus two hyperelliptic $\widehat{\wp}$ functions constructed from the σ function in the form:

$$\widehat{\wp}_{ij}(u_1, u_2) = -\frac{\partial^2 \log \sigma(u_1, u_2)}{\partial u_i \partial u_j}. \tag{8}$$

Though $\wp_{ij}(u_1, u_2)$ and $\widehat{\wp}_{ij}(u_1, u_2)$ satisfy the same integrability conditions:

$$\partial_i \wp_{jk}(u_1, u_2) = \partial_j \wp_{ik}(u_1, u_2) \quad \text{and} \quad \partial_i \widehat{\wp}_{jk}(u_1, u_2) = \partial_j \widehat{\wp}_{ik}(u_1, u_2),$$

$\wp_{ij}(u_1, u_2)$ and $\widehat{\wp}_{ij}(u_1, u_2)$ are not equal but differ by a constant. By the dimension analysis, we obtain $[\wp_{22}] = [1/u_2^2] = [y^2/x^4]$, $[\wp_{21}] = [1/u_1 u_2] = [y^2/x^3]$, $[\wp_{11}] = [1/u_1^2] = [y^2/x^2]$, $[a_4] = [y^2/x^4]$, $[a_3] = [y^2/x^3]$, $[a_2] = [y^2/x^2]$, so that three pairs $(\wp_{22}, a_4)$, $(\wp_{21}, a_3)$, and $(\wp_{11}, a_2)$ have the same dimensions. Thus, we put:

$$\wp_{22}(u_1, u_2) = \widehat{\wp}_{22}(u_1, u_2) - k_{22} a_4, \tag{9}$$

$$\wp_{21}(u_1, u_2) = \widehat{\wp}_{21}(u_1, u_2) - k_{21} a_3, \tag{10}$$

$$\wp_{11}(u_1, u_2) = \widehat{\wp}_{11}(u_1, u_2) - k_{11} a_2, \tag{11}$$

where a_i are coefficients of the hyperelliptic curve, and k_{22}, k_{21}, k_{11} are some numerical constants. We determine constants k_{ij} in such a way as whole differential equations transform covariantly [1].

Let us start with the following differential equations [27], [28],

$$1)\ \wp_{2222} - 6\wp_{22}^2 + 3\lambda_6 \wp_{11} - \lambda_5 \wp_{21} - \lambda_4 \wp_{22} - \frac{1}{8}\lambda_5 \lambda_3 + \frac{1}{2}\lambda_6 \lambda_2 = 0, \tag{12}$$

$$2)\ \wp_{2221} - 6\wp_{22}\wp_{21} + \frac{1}{2}\lambda_5 \wp_{11} - \lambda_4 \wp_{21} + \frac{1}{4}\lambda_6 \lambda_1 = 0, \tag{13}$$

$$3)\ \wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} - \frac{1}{2}\lambda_3 \wp_{21} + \frac{1}{2}\lambda_6 \lambda_0 = 0, \tag{14}$$

$$4)\ \wp_{2111} - 6\wp_{21}\wp_{11} - \lambda_2 \wp_{21} + \frac{1}{2}\lambda_1 \wp_{22} + \frac{1}{4}\lambda_5 \lambda_0 = 0, \tag{15}$$

$$5)\ \wp_{1111} - 6\wp_{11}^2 - \lambda_2 \wp_{11} - \lambda_1 \wp_{21} + 3\lambda_0 \wp_{22} - \frac{1}{8}\lambda_3 \lambda_1 + \frac{1}{2}\lambda_4 \lambda_0 = 0. \tag{16}$$

[1]In the general coordinate transformation, a tensor $T_{\mu\nu}$ transforms covariantly in the form $T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}$. We use the terminology transform covariantly in the same way. Observing a transformation $\widehat{\wp}'_{ij} = \frac{\partial u_p}{\partial u'_i}\frac{\partial u_q}{\partial u'_j}\widehat{\wp}_{pq}$ which shows up later, we may say that $\widehat{\wp}_{ij}$ transform covariantly.

We rewrite the above differential equations with a_n instead of λ_n. Later we will explain how to determine k_{ij}, but we provide here the values of them, $k_{22} = 3/2, k_{21} = 1/2, k_{11} = 3/2$. Thus, the constant shift of $\wp_{ij}$ are given in the form:

$$\wp_{22} = \widehat{\wp}_{22} - \frac{3}{2}a_4, \quad \wp_{21} = \widehat{\wp}_{21} - \frac{1}{2}a_3, \quad \wp_{11} = \widehat{\wp}_{11} - \frac{3}{2}a_2. \tag{17}$$

Hence, we obtain constant shifted differential equations [27]:

$$1)' \ \widehat{\wp}_{2222} - 6\widehat{\wp}_{22}^2 + 3\left(a_6\widehat{\wp}_{11} - 2a_5\widehat{\wp}_{21} + a_4\widehat{\wp}_{22}\right) + 3\left(a_6a_2 - 4a_5a_3 + 3a_4^2\right) = 0, \tag{18}$$

$$2)' \ \widehat{\wp}_{2221} - 6\widehat{\wp}_{22}\wp_{21} + 3\left(a_5\widehat{\wp}_{11} - 2a_4\widehat{\wp}_{21} + a_3\widehat{\wp}_{22}\right) + \frac{3}{2}\left(a_6a_1 - 3a_5a_2 + 2a_4a_3\right) = 0, \tag{19}$$

$$3)' \ \widehat{\wp}_{2211} - (4\widehat{\wp}_{21}^2 + 2\widehat{\wp}_{22}\widehat{\wp}_{11}) + 3\left(a_4\widehat{\wp}_{11} - 2a_3\widehat{\wp}_{21} + a_2\widehat{\wp}_{22}\right) + \frac{1}{2}\left(a_6a_0 - 9a_4a_2 + 8a_3^2\right) = 0, \tag{20}$$

$$4)' \ \widehat{\wp}_{2111} - 6\widehat{\wp}_{21}\widehat{\wp}_{11} + 3\left(a_3\widehat{\wp}_{11} - 2a_2\widehat{\wp}_{21} + a_1\widehat{\wp}_{22}\right) + \frac{3}{2}\left(a_5a_0 - 3a_4a_1 + 2a_3a_2\right) = 0, \tag{21}$$

$$5)' \ \widehat{\wp}_{1111} - 6\widehat{\wp}_{11}^2 + 3\left(a_2\widehat{\wp}_{11} - 2a_1\widehat{\wp}_{21} + a_0\widehat{\wp}_{22}\right) + 3\left(a_4a_0 - 4a_3a_1 + 3a_2^2\right) = 0. \tag{22}$$

From Eqs.(18)–(22), we define the first, second, third and fourth term of each equation as the component of vectors of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ in the form:

$$\mathbf{P} = \begin{pmatrix} \widehat{\wp}_{2222} \\ \widehat{\wp}_{2221} \\ \widehat{\wp}_{2211} \\ \widehat{\wp}_{2111} \\ \widehat{\wp}_{1111} \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} -6\widehat{\wp}_{22}^2 \\ -6\widehat{\wp}_{22}\widehat{\wp}_{21} \\ -4\widehat{\wp}_{21}^2 - 2\widehat{\wp}_{22}\widehat{\wp}_{11} \\ -6\widehat{\wp}_{21}\widehat{\wp}_{11} \\ -6\widehat{\wp}_{11}^2 \end{pmatrix}, \ \mathbf{R} = 3\begin{pmatrix} a_6\widehat{\wp}_{11} - 2a_5\widehat{\wp}_{21} + a_4\widehat{\wp}_{22} \\ a_5\widehat{\wp}_{11} - 2a_4\widehat{\wp}_{21} + a_3\widehat{\wp}_{22} \\ a_4\widehat{\wp}_{11} - 2a_3\widehat{\wp}_{21} + a_2\widehat{\wp}_{22} \\ a_3\widehat{\wp}_{11} - 2a_2\widehat{\wp}_{21} + a_1\widehat{\wp}_{22} \\ a_2\widehat{\wp}_{11} - 2a_1\widehat{\wp}_{21} + a_0\widehat{\wp}_{22} \end{pmatrix},$$

$$\mathbf{S} = 3\begin{pmatrix} a_6a_2 - 4a_5a_3 + 3a_4^2 \\ (a_6a_1 - 3a_5a_2 + 2a_4a_3)/2 \\ (a_6a_0 - 9a_4a_2 + 8a_3^2)/6 \\ (a_5a_0 - 3a_4a_1 + 2a_3a_2)/2 \\ a_4a_0 - 4a_3a_1 + 3a_2^2 \end{pmatrix}. \tag{23}$$

Each differential equation is given in the form:

$$P_n + Q_n + R_n + S_n = 0, \ (n = 1, 2, \cdots, 5), \tag{24}$$

which provide Eqs.(18)–(22).

Next, we consider the generalized dual transformation of the form:

$$x' = \frac{ax - c}{-bx + d}, \quad y' = \frac{y}{(-bx + d)^3}, \quad \text{with} \quad ad - bc = 1, \tag{25}$$

in such a way as such transformation makes the hyperelliptic curve on $\mathbf{R}$ to be invariant. Then a'_n $(n = 1, 2, \cdots, 6)$ are systematically determined from the relation:

$$\sum_{n=0}^{6} {}_6C_n a_n (bx' + a)^{6-n}(dx' + c)^n = \sum_{n=0}^{6} {}_6C_n a'_n x'^n. \tag{26}$$

The explicit expressions of the transformations of a_n are given in Appendix A. We put $a_6 = 0$ and $a'_6 = 0$ after the transformation. The transformed Jacobi's inversion relations are given by:

$$\mathrm{d}u'_1 = \frac{\mathrm{d}x'_1}{y'_1} + \frac{\mathrm{d}x'_2}{y'_2} = \frac{(-bx_1 + d)\mathrm{d}x_1}{y_1} + \frac{(-bx_2 + d)\mathrm{d}x_2}{y_2} = d \ \mathrm{d}u_1 - b \ \mathrm{d}u_2, \tag{27}$$

$$\mathrm{d}u'_2 = \frac{x'_1\mathrm{d}x'_1}{y'_1} + \frac{x'_2\mathrm{d}x'_2}{y'_2} = \frac{(ax_1 - c)\mathrm{d}x_1}{y_1} + \frac{(ax_2 - c)\mathrm{d}x_2}{y_2} = -c \ \mathrm{d}u_1 + a \ \mathrm{d}u_2. \tag{28}$$

Then we obtain:

$$\frac{\partial}{\partial u'_1} = a\frac{\partial}{\partial u_1} + c\frac{\partial}{\partial u_2}, \quad \frac{\partial}{\partial u'_2} = b\frac{\partial}{\partial u_1} + d\frac{\partial}{\partial u_2}. \tag{29}$$

We require that the hyperelliptic curve becomes invariant under the transformation, which implies that the σ function is invariant. From the invariance of the σ function under the transformation, $\sigma'(u'_1, u'_2) = \sigma(u_1, u_2)$, the transformed $\widehat{\wp}_{ij}$ functions are given by:

$$\widehat{\wp}'_{ij}(u'_1, u'_2) = -\frac{\partial^2 \log \sigma'(u_1, u_2)}{\partial u'_i \partial u'_i} = -\frac{\partial^2 \log \sigma(u_1, u_2)}{\partial u'_i \partial u'_i}.$$

Then $\widehat{\wp}_{ij}$ transform in covariant forms,

$$\widehat{\wp}'_{22} = \frac{\partial u_p}{\partial u'_2}\frac{\partial u_q}{\partial u'_2}\widehat{\wp}_{pq} = d^2\widehat{\wp}_{22} + 2bd\widehat{\wp}_{21} + b^2\widehat{\wp}_{11}, \tag{30}$$

$$\widehat{\wp}'_{21} = \frac{\partial u_p}{\partial u'_2}\frac{\partial u_q}{\partial u'_1}\widehat{\wp}_{pq} = cd\widehat{\wp}_{22} + (ad+bc)\widehat{\wp}_{21} + ab\widehat{\wp}_{11}, \tag{31}$$

$$\widehat{\wp}'_{11} = \frac{\partial u_p}{\partial u'_1}\frac{\partial u_q}{\partial u'_1}\widehat{\wp}_{pq} = c^2\widehat{\wp}_{22} + 2ac\widehat{\wp}_{21} + a^2\widehat{\wp}_{11}. \tag{32}$$

A simple rule to obtain the above result is as follows. From Eq.(29), we consider $P'_1 = aP_1 + cP_2$ and $P'_2 = bP_1 + dP_2$. Making ${P'_2}^2 = b^2P_1^2 + 2bdP_1P_2 + d^2P_2^2$ and replace

$${P'_2}^2 \to \widehat{\wp}'_{22}, \quad P_2^2 \to \widehat{\wp}_{22}, \quad P_2P_1 \to \widehat{\wp}_{21}, \quad P_1^2 \to \widehat{\wp}_{11},$$

which gives Eq.(30). This simplified rule is useful to obtain transformed expressions of $\widehat{\wp}'_{ijk\ell}$ by considering $P'_iP'_jP'_kP'_\ell$. Thus $\widehat{\wp}_{ijk\ell}$ transform in the covariant form:

$$\widehat{\wp}'_{ijk\ell} = \frac{\partial u_p}{\partial u'_i}\frac{\partial u_q}{\partial u'_j}\frac{\partial u_r}{\partial u'_k}\frac{\partial u_s}{\partial u'_\ell}\widehat{\wp}_{pqrs}, \tag{33}$$

which provides

$$\begin{pmatrix}\widehat{\wp}'_{2222}\\ \widehat{\wp}'_{2221}\\ \widehat{\wp}'_{2211}\\ \widehat{\wp}'_{2111}\\ \widehat{\wp}'_{1111}\end{pmatrix} = \begin{pmatrix} d^4 & 4bd^3 & 6b^2d^2 & 4b^3d & b^4\\ cd^3 & (ad+3bc)d^2 & 3(ad+bc)bd & (3ad+bc)b^2 & ab^3\\ c^2d^2 & 2(ad+bc)cd & a^2d^2+4abcd+b^2c^2 & 2(ad+bc)ab & a^2b^2\\ c^3d & (3ad+bc)c^2 & 3(ad+bc)ac & (ad+3bc)a^2 & a^3b\\ c^4 & 4ac^3 & 6a^2c^2 & 4a^3c & a^4\end{pmatrix}\begin{pmatrix}\widehat{\wp}_{2222}\\ \widehat{\wp}_{2221}\\ \widehat{\wp}_{2211}\\ \widehat{\wp}_{2111}\\ \widehat{\wp}_{1111}\end{pmatrix}. \tag{34}$$

We denote this as $\mathbf{P}' = M\mathbf{P}$. Then, by the same M, we can prove $\mathbf{Q}' = M\mathbf{Q}$. We determined k_{ij} in Eqs.(9)–(11) in such a way as $\mathbf{R}$ transform in the same way as $\mathbf{R}' = M\mathbf{R}$, so that $k_{22} = 3/2$, $k_{21} = 1/2$ and $k_{11} = 3/2$ can be obtained. Thus, as we have promised, the constant shift of $\wp_{ij}$ has been determined as Eq.(17). Hence, by using M and k_{ij}, we can prove $\mathbf{S}' = M\mathbf{S}$. We define the total vector $\mathbf{T} = \mathbf{P} + \mathbf{Q} + \mathbf{R} + \mathbf{S}$, whole differential equations transform covariantly in the form $\mathbf{T}' = M\mathbf{T}$. From differential equations $\mathbf{T} = 0$, we obtain $\mathbf{T}' = 0$, that is, the set of differential equations $\mathbf{T} = 0$ is invariant.

We have shown that differential equations have the Lie group (continuous group) structure; yet an issue is what type of Lie group structure the differential equations have. To elucidate the problem, we try to find quadratic invariances defined from some vector $\mathbf{X}$. We here adopt $\mathbf{X} = \mathbf{P}$, that is,

$$X_1 = \widehat{\wp}_{2222}, \quad X_2 = \widehat{\wp}_{2221}, \quad X_3 = \widehat{\wp}_{2211}, \quad X_4 = \widehat{\wp}_{2111}, \quad X_5 = \widehat{\wp}_{1111}.$$

Since the dual transformation is a special case of the generalized transformation, the invariance of the following dual transformations:

$$\widehat{\wp}_{2222} \leftrightarrow \widehat{\wp}_{1111}, \quad \widehat{\wp}_{2221} \leftrightarrow \widehat{\wp}_{2111}, \quad \widehat{\wp}_{2211} \leftrightarrow \widehat{\wp}_{2211}$$

are necessary. Thus, as the quadratic invariance, we obtain:

$$I = \ell_1\widehat{\wp}_{2222}\widehat{\wp}_{1111} + \ell_2\widehat{\wp}_{2221}\widehat{\wp}_{2111} + \ell_3\widehat{\wp}_{2211}^2 = \ell_1X_1X_5 + \ell_2X_2X_4 + \ell_3X_3^2. \tag{35}$$

By imposing the invariance under the transformation, the coefficients ℓ_1, ℓ_2, ℓ_3 are determined to be:

$$I = \widehat{\wp}_{2222}\widehat{\wp}_{1111} - 4\widehat{\wp}_{2221}\widehat{\wp}_{2111} + 3\widehat{\wp}_{2211}^2 = X_1X_5 - 4X_2X_4 + 3X_3^2 = \text{inv.}. \tag{36}$$

Even if we adopt $\mathbf{X}$ as any of $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}\}$, Eq.(36) gives invariants. Thus, we define:

$$X_1 = Y_1 + Y_4, \quad X_2 = \frac{Y_5 + Y_2}{2}, \quad X_3 = \frac{Y_3}{\sqrt{3}}, \quad X_4 = \frac{Y_5 - Y_2}{2}, \quad X_5 = Y_1 - Y_4, \tag{37}$$

and we arrive at the quadratic invariance of the form:

$$I = \left(Y_1^2 + Y_2^2 + Y_3^2\right) - \left(Y_4^2 + Y_5^2\right) = \text{inv.} \tag{38}$$

Therefore, we conclude that differential equations have the $\mathrm{SO}(3,2) \cong$Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie group structure, which is consistent with our previous results[24], [25]. At this level, we put $a_6 = 0$ and $a'_6 = 0$, and $a'_6 = 0$ is realized by taking the standard form of the hyperelliptic curve with $a_0 = -(6a_5d^5 + 15a_4bd^4 + 20a_3b^2d^3 + 15a_2b^3d^2 + 6a_1b^4d)/b^5$. As the invariance of the transformation is identically satisfied, we obtain the same result even if we put constraints $a_6 = 0$ and $a'_6 = 0$.

A. The Lie Algebraic Approach to the Constant Shift of $\wp_{ij}$ and the Quadratic Invariant

We consider the following three infinitesimal transformations derived from Eq.(25) [29],

$$\text{i) } x' = x + \epsilon, \qquad (a = 1,\ b = 0,\ c = -\epsilon,\ d = 1) \tag{39}$$
$$\text{ii) } x' = x + \epsilon x, \qquad (a = 1 + \epsilon/2,\ b = 0,\ c = 0,\ d = 1 - \epsilon/2) \tag{40}$$
$$\text{iii) } x' = x + \epsilon x^2, \quad (a = 1,\ b = \epsilon,\ c = 0,\ d = 1) \tag{41}$$

where ϵ is an infinitesimal parameter. Denoting $\delta x = x' - x$, each infinitesimal transformations are represented by generators Q_1, Q_2 and Q_3 as follows:

$$\text{i) } \delta x = \epsilon = [\epsilon Q_1, x], \qquad Q_1 = \frac{\partial}{\partial x} \tag{42}$$
$$\text{ii) } \delta x = \epsilon x = [\epsilon Q_2, x], \quad Q_2 = x\frac{\partial}{\partial x} \tag{43}$$
$$\text{iii) } \delta x = \epsilon x^2 = [\epsilon Q_3, x], \quad Q_3 = x^2\frac{\partial}{\partial x} \tag{44}$$

Commutation relations $[Q_3, Q_2] = -Q_3$, $[Q_1, Q_2] = Q_1$, $[Q_3, Q_1] = -2Q_2$ can be modified into the following form:

$$[iQ_3, Q_2] = -iQ_3, \quad [iQ_1, Q_2] = iQ_1, \quad [iQ_3, iQ_1] = 2Q_2. \tag{45}$$

On the other hand, the Lie algebra of SO(3), $[J_a, J_b] = i\epsilon_{abc}J_c$, can be rewritten in the form:

$$[J_+, J_3] = -J_+, \quad [J_-, J_3] = J_-, \quad [J_+, J_-] = 2J_3,$$

with $J_\pm = J_1 \pm iJ_2$. Then we have the correspondence:

$$J_+ \leftrightarrow iQ_3, \quad J_- \leftrightarrow iQ_1, \quad J_3 \leftrightarrow Q_2,$$

which gives the SO(2,1) Lie algebra structure.

For our purpose to fix k_{ij} which are coefficients of the constant shift of $\wp_{ij}$, and ℓ_i which are the coefficients of quadratic invariance, it is sufficient to consider the infinitesimal transformation i):

$$x_1' = x_1 + \epsilon, \quad x_2' = x_2 + \epsilon, \quad y_1' = y_1, \quad y_2' = y_2. \tag{46}$$

In this case, a_i transform as:

$$\begin{aligned} &a_6' = a_6, \quad a_5' = a_5 - \epsilon a_6, \quad a_4' = a_4 - 2\epsilon a_5, \quad a_3' = a_3 - 3\epsilon a_4, \\ &a_2' = a_2 - 4\epsilon a_3, \quad a_1' = a_1 - 5\epsilon a_2, \quad a_0' = a_0 - 6\epsilon a_1. \end{aligned} \tag{47}$$

Transformation laws of $\widehat{\wp}_{ij}$ and $\widehat{\wp}_{ijkl}$ are determined in the following way:

$$\widehat{\wp}_{22}' = \widehat{\wp}_{22}, \quad \widehat{\wp}_{21}' = \widehat{\wp}_{21} - \epsilon\widehat{\wp}_{22}, \quad \widehat{\wp}_{11}' = \widehat{\wp}_{11} - 2\epsilon\widehat{\wp}_{21}, \tag{48}$$

and

$$\begin{aligned} &\widehat{\wp}_{2222}' = \widehat{\wp}_{2222}, \quad \widehat{\wp}_{2221}' = \widehat{\wp}_{2221} - \epsilon\widehat{\wp}_{2222}, \quad \widehat{\wp}_{2211}' = \widehat{\wp}_{2211} - 2\epsilon\widehat{\wp}_{2221}, \\ &\widehat{\wp}_{2111}' = \widehat{\wp}_{2111} - 3\epsilon\widehat{\wp}_{2211}, \quad \widehat{\wp}_{1111}' = \widehat{\wp}_{1111} - 4\epsilon\widehat{\wp}_{2111}. \end{aligned} \tag{49}$$

We put $a_6 = a_6' = 0$ after the transformation.

In order to determine k_{ij}, we can use the infinitesimal transformation of differential equations. The infinitesimal transformation of Eq.(47) raise the order of differential equations, that is, we obtain Eq.(21), Eq.(20), Eq.(19), and Eq.(18) from Eq.(22), Eq.(21), Eq.(20), and Eq.(19), respectively. Using this method, we reproduce Eq.(17) [29]. Here, we demonstrate another method to use the fundamental relation, which generate all differential equation by differentiation, of the form:

$$\wp_{22}(u_1, u_2)x_1x_2 + \wp_{21}(u_1, u_2)(x_1 + x_2) + \wp_{11}(u_1, u_2) = \frac{F(x_1, x_2) - 2y_1y_2}{4(x_1 - x_2)^2}. \tag{50}$$

This relation is trivially satisfied by using:

$$\wp_{22}(u_1, u_2) = \frac{\lambda_5}{4}(x_1 + x_2), \quad \wp_{21}(u_1, u_2) = -\frac{\lambda_5}{4}x_1x_2, \quad \wp_{11}(u_1, u_2) = \frac{F(x_1, x_2) - 2y_1y_2}{4(x_1 - x_2)^2}.$$

By using the shifted $\wp_{ij}$ functions of Eqs.(9)–(11), Eq.(50) becomes in the form:

$$\begin{aligned} &(\widehat{\wp}_{22}(u_1, u_2) - k_{22}a_4)x_1x_2 + (\widehat{\wp}_{21}(u_1, u_2) - k_{21}a_3)(x_1 + x_2) + (\widehat{\wp}_{11}(u_1, u_2) - k_{11}a_2) \\ &= \frac{F(x_1, x_2) - 2y_1y_2}{4(x_1 - x_2)^2}. \end{aligned} \tag{51}$$

Coefficients k_{ij} are determined by the invariance of this fundamental relation under the infinitesimal transformation as follows:

$$k_{22} = \frac{3}{2}, \quad k_{21} = \frac{1}{2}, \quad k_{11} = \frac{3}{2},$$

which reproduces Eq.(17). This is the necessary condition that the fundamental relation is invariant under the finite transformation.

Next, we determine ℓ_i by using Eq.(35). By imposing the invariance of Eq.(35) under the infinitesimal transformation, we obtain:

$$\begin{aligned} I \to I' \quad &= \ell_1 X_1(X_5 - 4\epsilon X_4) + \ell_2(X_2 - \epsilon X_1)(X_4 - 3\epsilon X_3) + \ell_3(X_3 - 2\epsilon X_2)^2 \\ &= I - \epsilon((4\ell_1 + \ell_2)X_1X_4 + (3\ell_2 + 4\ell_3)X_2X_3) = I, \end{aligned}$$

which gives $\ell_1 : \ell_2 : \ell_3 = 1 : -4 : 3$. This is the necessary condition that I is invariant under the finite transformation. Thus, we obtain:

$$I = X_1X_5 - 4X_2X_4 + 3X_3^2 = \text{inv.}, \tag{52}$$

which reproduces Eq.(36).

III. Conclusion

In the previous study, by directly using differential equations of genus two hyperelliptic $\wp$ functions, we demonstrated that the half-period addition formula for genus two hyperelliptic $\wp$ functions provides the order two Sp(4,$\mathbf{R}$) Lie group structure. In this study, we have considered the generalized dual transformation for hyperelliptic $\wp$ functions. By the constant shift of $\wp_{ij}$ functions, we have deduced that differential equations transform covariantly under such transformation. By constructing the quadratic invariance under such transformation, we have shown that hyperelliptic $\wp$ functions possess the SO(3,2) $\cong$ Sp(4,$\mathbf{R}$)/$\mathbf{Z_2}$ Lie group structure for genus two case.

Appendix A
The Transformation of Coefficients a_n $(n = 0, 1, \cdots, 6)$ in the Genus Two Hyperelliptic Curve

$$1)\ a_6' = a_6d^6 + 6a_5bd^5 + 15a_4b^2d^4 + 20a_3b^3d^3 + 15a_2b^4d^2 + 6a_1b^5d + a_0b^6, \tag{53}$$

$$\begin{aligned} 2)\ a_5' = \ & a_6cd^5 + a_5(ad + 5bc)d^4 + 5a_4(ad + 2bc)bd^3 + 10a_3(ad + bc)b^2d^2 \\ & +5a_2(2ad + bc)b^3d + a_1(5ad + bc)b^4 + a_0ab^5 \end{aligned} \tag{54}$$

$$\begin{aligned} 3)\ a_4' = \ & a_6c^2d^4 + 2a_5(ad + 2bc)cd^3 + a_4(a^2d^2 + 8abcd + 6b^2c^2)d^2 + 4a_3(a^2d^2 + 3abcd + b^2c^2)bd \\ & +a_2(6a^2d^2 + 8abcd + b^2c^2)b^2 + 2a_1(2ad + bc)ab^3 + a_0a^2b^4 \end{aligned} \tag{55}$$

$$\begin{aligned} 4)\ a_3' = \ & a_6c^3d^3 + 3a_5(ad + bc)c^2d^2 + 3a_4(a^2d^2 + 3abcd + b^2c^2)cd \\ & +a_3(a^3d^3 + 9a^2bcd^2 + 9ab^2c^2d + b^3c^3) + 3a_2(a^2d^2 + 3abcd + b^2c^2)ab \\ & +3a_1(ad + bc)a^2b^2 + a_0a^3b^3 \end{aligned} \tag{56}$$

$$\begin{aligned} 5)\ a_2' = \ & a_6c^4d^2 + 2a_5(2ad + bc)c^3d + a_4(6a^2d^2 + 8abcd + b^2c^2)c^2 + 4a_3(a^2d^2 + 3abcd + b^2c^2)ac \\ & +a_2(a^2d^2 + 8abcd + 6b^2c^2)a^2 + 2a_1(ad + 2bc)a^3b + a_0a^4b^2 \end{aligned} \tag{57}$$

$$\begin{aligned} 6)\ a_1' = \ & a_6c^5d + a_5(5ad + bc)c^4 + 5a_4(2ad + bc)ac^3 + 10a_3(ad + bc)a^2c^2 \\ & +5a_2(ad + 2bc)a^3c + a_1(ad + 5bc)a^4 + a_0a^5b \end{aligned} \tag{58}$$

$$7)\ a_0' = a_6c^6 + 6a_5ac^5 + 15a_4a^2c^4 + 20a_3a^3c^3 + 15a_2a^4c^2 + 6a_1a^5c + a_0a^6. \tag{59}$$

We put $a_6 = 0$ and $a_6' = 0$ after the transformation. For example, we can realize $a_6' = 0$ by taking the standard form of the hyperelliptic curve with $a_0 = -(6a_5d^5 + 15a_4bd^4 + 20a_3b^2d^3 + 15a_2b^3d^2 + 6a_1b^4d)/b^5$.

References

[1] Gardner C.S, Greene J.M, Kruskal M.D, Miura R.M. Method for Solving the Korteweg-de Vries Equation. *Phys. Rev. Lett.*, 1967; 19: 1095-1097.
[2] Lax P.D. Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Commun. Pure and Appl. Math.*, 1968; 21: 467-490.
[3] Zakharov V.E, Shabat A.B. Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media. *Sov. Phys. JETP*, 1972; 34: 62-69.
[4] Ablowitz M.J, Kaup D.J, Newell A.C, Segur H. Nonlinear-evolution Equations of Physical Significance. *Phys. Rev. Lett.*, 1973; 31: 125-127.
[5] Bianchi L. Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante. *Ann. Scuola Norm. Sup. Pisa (1)*, 1879; 2: 285-341.
[6] Hermann R. Pseudopotentials of Estabrook and Wahlquist, the Geometry of Solitons, and the Theory of Connections. *Phys. Rev. Lett.*, 1976; 36: 835-836.
[7] Sasaki R. Soliton Equation and Pseudospherical Surfaces. *Nucl. Phys.*, 1979; B154: 343-357.
[8] Wahlquist H.D, and Estabrook F.B. Bäcklund Transformation for Solution of Korteweg-de Vries Equation. *Phys. Rev. Lett.*, 1973; 31: 1386-1390.
[9] Wadati M. Bäcklund Transformation for Solutions of the Modified Korteweg-de Vries Equation. *J. Phys. Soc. Jpn.*, 1974; 36: 1498.
[10] Konno K, Wadati M. Simple Derivation of Backlund Transformation from Riccati Form of Inverse Method. *Prog. Theor. Phys.*, 1975; 53: 1652-1656.

[11] Hirota R. Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys. Rev. Lett.*, 1971; 27: 1192-1194.
[12] Hirota R. Exact Solution of the Modified Korteweg-de Vries Equation for Multiple Collisions of Solitons. *J. Phys. Soc. Jpn.*, 1972; 33: 1456-1458.
[13] Sato M. Soliton Equations as Dynamical Systems on an Infinite Dimensional Grassmann Manifolds. *RIMS Kokyuroku (Kyoto University)*, 1981; 439: 30-46.
[14] Miwa T, Jimbo M, Date E. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*, Cambridge University Press, Cambridge, 2000.
[15] Date E, Kashiwara M, Miwa T. Vertex Operators and τ Functions: Transformation Groups for Soliton Equations. II. *Proc. Japan Acad.*, 1981; 57A: 387-392.
[16] Jimbo M, Miwa T. Solitons and Infinite Dimensional Lie Algebra. *Publ. RIMS. Kyoto Univ.*, 1983; 19: 943-1001.
[17] Weiss J. The Painlevé Property for Partial Differential Equations. II: Bäcklund Transformation, Lax Pairs, and the Schwarzian Derivative. *J. Math. Phys.*, 1983; 24: 1405-1413.
[18] Hayashi M, Shigemoto K, Tsukioka T. The Construction of the mKdV Cyclic Symmetric N-soliton Solution by the Bäcklund Transformation. *Mod. Phys. Lett.*, 2019; A34: 1950136.
[19] Hayashi M, Shigemoto K, Tsukioka T. The Static Elliptic N-soliton Solutions of the KdV Equation. *J. Phys. Commun.*, 2019; 3: 045004.
[20] Hayashi M, Shigemoto K, Tsukioka T. The Unified Soliton System as the AdS_2 System. *J. Phys. Commun.*, 2019; 3: 085015.
[21] Hayashi M, Shigemoto K, Tsukioka T. Common Hirota Form Bäcklund Transformation for the Unified Soliton System. *J. Phys. Commun.*, 2020; 4: 015014.
[22] Hayashi M, Shigemoto K, Tsukioka T. Elliptic Solutions for Higher Order KdV Equations. *J. Phys. Commun.*, 2020; 4: 045013.
[23] Hayashi M, Shigemoto K, Tsukioka T. Differential Equations of Genus Four Hyperelliptic $\wp$ Functions. *J. Phys. Commun.*, 2021; 5: 105008.
[24] Hayashi M, Shigemoto K, Tsukioka T. Two Flows Kowalevski Top as the Full Genus Two Jacobi's Inversion Problem and Sp(4,R) Lie Group Structure. *J. Phys. Commun.*, 2022; 6: 025006.
[25] Hayashi M, Shigemoto K, Tsukioka T. The Half-period Addition Formulae for Genus Two Hyperelliptic $\wp$ Functions and the Sp(4,R) Lie Group Structure. *J. Phys. Commun.*, 2022; 6: 085004.
[26] Baker H.F. *Abelian Functions: Abel's theorem and the allied theory of theta functions*, Cambridge University Press, Cambridge, 1995: p.321-p.323, p.328.
[27] Baker H.F. *An Introduction To the Theory of Multiply Periodic Functions*, Cambridge University Press, Cambridge, 1909: p.38, p.49.
[28] Baker H.F. On a Certain System of Differential Equations Defining Periodic Functions. *Proc. Camb. Phil. Soc.* , 1898; 9: 513-520.
[29] Baker H.F. On the Differential Equations of the Hyperelliptic Functions. *Proc. Camb. Phil. Soc.*, 1903; 12: 219-239.