On Reaction-Diffusion Equation with Generalized Composite Fractional Derivative

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***Abstract* —In this paper, we obtain the Sumudu transform of generalized composite fractional derivative and some lemmas related to inverse Sumudu transform. Further, we find solution of nonlinear reaction diffusion equation with generalized composite fractional derivative by applying the Sumudu and Fourier transforms.**

***Keywords* — Fourier transform, Fractional derivative, Reaction diffusion equation, Sumudu transform.**

I. INTRODUCTION

Reaction diffusion equations have found applications in various branches of science and technology [1]-[3]. The classical reaction diffusion equation is given by [4]

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + \theta \cdot X(N), \tag{1}$$

where D is diffusion coefficient and $X(N)$ a nonlinear function representing reaction kinetics.

A generalization of (1) was proposed by [5] and is given as

$$\frac{\partial^2 N}{\partial t^2} + \xi \cdot \frac{\partial N}{\partial t} = \chi^2 \frac{\partial^2 N}{\partial x^2} + \eta^2 N(x,t). \tag{2}$$

Further, [6] gave another generalization of reaction diffusion equation with the fractional derivatives as

$${}_0\mathcal{D}_t^{\alpha} N(x,t) + \xi \cdot {}_0\mathcal{D}_t^{\beta} N(x,t) = \chi^2 {}_{-\infty}\mathcal{D}_x^{\gamma} N(x,t) + \eta^2 N(x,t) + \psi(x,t), \tag{3}$$

where $\alpha > \beta$, $\psi(x,t)$ represents the nonlinearity in the system and ξ demonstrates the nonlinearity of the system. Recently, several authors have studied reaction diffusion equation with fractional deriavtives [7]-[9].

The fractional derivatives are continuously showing their potential in the modeling of real world problems. This is leading to continuous development of fractional derivatives. Reference [10] defined a fractional derivative which was composite of Riemann-Liouville and caputo fractional derivative of same order [11], it thus possessed properties of both the derivatives with additional advantage due to their composition. Reference [12] further defined a composition of these derivatives, allowing different fractional orders of Riemann-Liouville and Caputo fractional derivatives, thus widening the application of this generalized composite fractional derivative (GCFD). In this paper, we investigate reaction diffusion equation with GCFD.

Integral transforms such as Laplace, Fourier, Mellin, Hankel, Sumudu etc. which are being extensively applied in several branches of science and technology [13]-[15]. Besides, Sumudu transform (analogous to Laplace transform) was proposed by [16] in early 1990’s with the motivational superiority over other integral transforms, mostly the scale and unity preserving features that could yield adequacy when solving differential equations. Moreover, it is notable that Sumudu transformation method provides the solution in closed form and possess the capability to scale down the volume of computional work in contrast to the other methods. Several properties and applications related to Sumudu transform can be seen in the literature [17]-[19].

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II. Definitions and Preliminaries

Definition II. 1 ([11]) The Riemann-Liouville integral operator of order $\alpha > 0$ of a function $\psi(t)$ is defined as:

$$ {}_0\mathcal{J}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\psi(u)du, \;\; \alpha \in \mathbb{C} \text{ and } t > 0. \tag{4}$$

Definition II. 2 ([11]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $\psi(t)$ is defined as:

$$ {}_0\mathcal{D}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_0^t (t-u)^{k-\alpha-1}\psi(u)du, k-1 < \alpha < k, k \in \mathbb{N}. \tag{5}$$

Definition II. 3 ([11]) The Caputo fractional derivative of order $\alpha > 0$ of a function $\psi(t)$ is defined as:

$$ {}_0^C\mathcal{D}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(k-\alpha)}\int_0^t (t-u)^{k-\alpha-1}\psi^{(k)}(u)du, \;\; k-1 < \alpha < k, k \in \mathbb{N}. \tag{6}$$

Definition II. 4 ([6], [20]) The Weyl fractional differential operator of order $\alpha > 0$ is defined as:

$$ {}_{-\infty}\mathcal{D}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_{-\infty}^t (t-u)^{k-\alpha-1}\psi(u)du, k-1 < \alpha < k, k \in \mathbb{N}. \tag{7}$$

The modified Fourier transform of the operator (7) given by [21], is as follows:

$$ F\{{}_{-\infty}\mathcal{D}_t^{\alpha}\psi(t)\} = -|p|^{\alpha}\psi^*(p). \tag{8}$$

where the Fourier transform is defined by the integral equation

$$ \psi^*(p) = \int_{-\infty}^{\infty}\psi(p)\exp(ipy)dy. $$

Definition II. 5 ([10]) For $0 < \alpha \leq 1$, and $0 \leq \beta \leq 1$, the Hilfer fractional derivative of order α and type β of a function $\psi(t)$ is given as:

$$ {}_0\mathcal{D}_t^{\alpha,\beta}\psi(t) = ({}_0\mathcal{J}_t^{\beta(1-\alpha)}\frac{d}{dt}({}_0\mathcal{J}_t^{(1-\beta)(1-\alpha)}\psi(t))). \tag{9}$$

Definition II. 6 ([12]) If $k-1 < \alpha, \beta \leq k$; $0 \leq \nu \leq 1$ and $k \in \mathbb{N}$, then the generalized composite fractional derivative (GCFD) of $\psi(t)$ is defined as:

$$ {}_0\mathcal{D}_t^{\alpha,\beta;\nu}\psi(t) = ({}_0\mathcal{J}_t^{\nu(k-\beta)}\frac{d^k}{dt^k}({}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi(t))). \tag{10}$$

The GCFD (10) for $\nu = 0$ and $\nu = 1$ reduces to Riemann-Liouville type fractional derivative of order α (5) and Caputo type fractional derivative of order β (6) respectively. Also, for $\alpha = \beta$, the GCFD (10) becomes Hilfer's fractional derivative of order α and type ν (9).

Definition II. 7 [22] A generalization of Mittag-Leffler function $E_{\alpha,\beta}(t)$, is given as:

$$ E_{\alpha,\beta}^{\gamma}(t) = \sum_{j=0}^{\infty}\frac{(\gamma)_j}{\Gamma(\alpha j+\beta)}\frac{t^j}{j!}, $$

where $\alpha > 0, \beta > 0, \alpha, \beta, t \in \mathbb{R}$ and $(\gamma)_j$ denotes shifted factorial and is defined as

$$ (\gamma)_0 = 1, (\gamma)_j = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+j-1), j = 1,2,\cdots, \gamma \neq 0. $$

Definition II. 8 The Sumudu transform denoted by $G(s)$ for a function $\psi(t)$ which was introduced by [16] is given as:

$$ G(s) = S[\psi(t), s] = \int_0^{\infty}\exp(-t)\psi(st)dt = \frac{1}{s}\int_0^{\infty}\exp(-\frac{t}{s})\psi(t)dt, \quad s \in (-\lambda_1, \lambda_2), $$

over the set of functions

$$\mathcal{A}=\left\{\psi(t)|\exists M,\lambda_1,\lambda_2>0,|\psi(t)|<M\exp(\tfrac{|t|}{\lambda_j}), if\ t\in(-1)^j\times[0,\infty)\right\}.$$

Following are some important properties of the Sumudu transform that shall be required in the upcoming sections.

Proposition II. 1 ([18], [23]) If M(s) and N(s) be the Sumudu transform of the functions ψ(t) and ϕ(t) respectively, then the Sumudu of their convolution is given as

$$S[(\psi(t)\star\phi(t)),s]=sM(s)N(s),$$

or equivalently,

$$S^{-1}[sM(s)N(s),t] \quad =(\psi(t)\star\phi(t)). \tag{11}$$

where,

$$(\psi(t)\star\phi(t))=\int_0^t\psi(u)\phi(t-u)du.$$

Theorem II. 1 ([23]) [Sumudu transform of fractional Integral]: Let the sumudu transform of the function ψ(t) be G(s), then the sumudu transform G(s) of fractional integral of ψ(t) of order α, is given as:

$$S[{}_0\mathcal{J}_t^\alpha\psi(t),s]=s^\alpha G(s),\ \ Re(\alpha)>0.$$

Theorem II. 2 ([23]) Suppose G(s) is the Sumudu transform of ψ(t), then the Sumudu of the m^{th} derivative $\psi^{(m)}(t)$ is denoted by $G_m(s)$ and is given as:

$$G_m(s)=S[\psi^m(t),s]=s^{-m}G(s)-\textstyle\sum_{j=0}^{m-1}s^{j-m}\psi^{(j)}(0),\ \ m\geq 1.$$

Theorem II. 3 ([23]) [Sumudu of fractional derivatives]: Let $k-1\leq\alpha<k$, $k\in\mathbf{N}$ and G(s) be the Sumudu of ψ(t), then the Sumudu $G_\alpha(s)$ of the Riemann-Liouville and Caputo fractional derivatives of order α, $\mathcal{D}^\alpha\psi(t)$ are given as:

$$G_\alpha(s)=S[{}_0\mathcal{D}_t^\alpha\psi(t),s]=s^{-\alpha}G(s)-\textstyle\sum_{j=0}^{k-1}s^{-(j+1)}[\mathcal{D}^{\alpha-j-1}\psi(t)|_{t=0}].$$

and

$$G_\alpha(s)=S[{}_0^C\mathcal{D}_t^\alpha\psi(t),s]=s^{-\alpha}G(s)-\sum_{j=0}^{k-1}s^{-(\alpha-j)}[\mathcal{D}^k\psi(t)|_{t=0}].$$

Theorem II. 4 ([7]) In the complex plane ℂ, for any Re(β) > 0, Re(γ) > 0 and ϑ ∈ ℂ, the following equality holds for inverse Sumudu transform:

$$S^{-1}[s^{\gamma-1}(1-\vartheta s^\beta)^{-\delta}]=t^{\gamma-1}E_{\beta,\gamma}^\delta(\vartheta t^\beta). \tag{12}$$

Theorem II. 5 ([7]) If $k-1<\alpha,\beta<k$ such that k ∈ ℕ and β, then the following equality holds

$$S^{-1}\left[\frac{1}{s(s^{-\alpha}+\xi\cdot s^{-\beta}+\rho)}\right]=\textstyle\sum_{r=0}^\infty(-\rho)^r t^{\alpha(r+1)-1}\times E_{\alpha-\beta,\alpha(r+1)}^{r+1}(-\xi\cdot t^{\alpha-\beta}). \tag{13}$$

III. Main Results

Before we give the Sumudu transform of GCFD, we give the following lemma

Lemma III. 1 If $k-1<\alpha,\beta;\ \mu,\delta<k$ such that k ∈ ℕ, 0 ≤ ν ≤ 1 and $\{\alpha-\nu(\alpha-\beta)\}>\{\mu-\nu(\mu-\delta)\}$, then the following equality holds

$$\begin{aligned}S^{-1}\left[\frac{s^{\nu(k-\beta)-k}}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}+\rho}\right]&=\textstyle\sum_{r=0}^\infty(-\rho)^r t^{\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k}\\&\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k+1}^{r+1}(-\xi\cdot t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)}).\end{aligned} \tag{14}$$

Proof. To prove (14), consider

$$\frac{s^{\nu(k-\beta)-k}}{(s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}+\rho)}$$
$$=\frac{s^{\nu(k-\beta)-k}}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}}(1+\frac{\rho}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}})^{-1}$$
$$=\sum_{r=0}^{\infty}(-\rho)^r s^{s^{\nu(k-\beta)-k}-\{\nu(\alpha-\beta)-\alpha\}(r+1)}(1+\xi\cdot s^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})^{-(r+1)} \quad (15)$$

Now applying the inverse Sumudu transform on both sides of (15) and then using (12), we obtain (14).

Theorem III.2 If G(s) is the Sumudu transform of ψ(t) and $k-1<\alpha,\beta\le k$ with $k\in\mathbb{N}$, then the Sumudu of the GCFD ${}_0\mathcal{D}_t^{\alpha,\beta;\nu}\psi(t)$ is given by

$$S[{}_0\mathcal{D}_t^{\alpha,\beta;\nu}\psi(t),s=s^{\nu(\alpha-\beta)-\alpha}G(s)-\sum_{j=0}^{k-1}s^{\nu(k-\beta)-k+j}\left[\left(\mathcal{D}^j({}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi)(t)\right)|_{t=0}\right. . \quad (16)$$

Proof. For simplicity, suppose $\phi(t)=\mathcal{D}^k{}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi(t)=\mathcal{D}^k\Phi(t)$. So, by (9) and using Theorem II.1, we have

$$S[{}_0\mathcal{D}_t^{\alpha,\beta;\nu}\psi(t),s]=S[{}_0\mathcal{J}_t^{\nu(k-\beta)}\phi(t),s]=s^{\nu(k-\beta)}S[\phi(t),s]=s^{\nu(k-\beta)}S[\mathcal{D}^k\Phi(t),s],$$

where $\Phi(t)={}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi(t)$. By applying Theorem II.2, we obtain

$$\begin{aligned}S[{}_0\mathcal{D}_t^{\alpha,\beta;\nu}\psi(t),s]=\;&s^{\nu(k-\beta)-k}S[{}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi(t),s]\\&-\sum_{j=0}^{k-1}s^{\nu(k-\beta)-k+j}[(\mathcal{D}^j({}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}\psi)(t))|_{t=0}].\end{aligned} \quad (17)$$

Again using Theorem II.1 in (17), we get the desired result (16).

Now, we solve reaction-diffusion equation with GCFD by using Sumudu and Fourier transformation

Theorem III.3 Consider the fractional diffusion equation

$${}_0\mathcal{D}_t^{\alpha,\beta;\nu}N(x,t)+\xi\cdot{}_0\mathcal{D}_t^{\mu,\delta;\nu}N(x,t)=\chi^2{}_{-\infty}\mathcal{D}_x^{\gamma}N(x,t)+\eta^2N(x,t)+\psi(x,t), \quad (18)$$

with the initial conditions

$$\left.\begin{aligned}\mathcal{D}^j{}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}N(x,t)|_{t=0}&=f_j(x)\\ \mathcal{D}^j{}_0\mathcal{J}_t^{(1-\nu)(k-\mu)}N(x,t)|_{t=0}&=g_j(x)\end{aligned}\right\};\quad j=0,1,2,\dots k-1,\&\,k\in\mathbb{N}, \quad (19)$$

where $k-1<\alpha,\beta;\ \mu,\delta\le k$, such that $\alpha>\mu,\beta>\delta$ and $0\le\nu\le1$. *Also*, χ is a diffusion coefficient, η is a constant which represents nonlinearity of the system and ψ is a nonlinear function for reaction kinetics, then the solution of (18) corresponding to $N(x,t)$ is as follow

$$\begin{aligned}N(x,t)=&\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k+j}f_j^*(p)\exp(-ipx)\\&\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k+j+1}^{r+1}(-\xi t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})\}dp\\&+\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\xi\cdot\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{k(\nu-1)+(r+1)(\alpha-\nu(\alpha-\beta))-\nu\delta+j}g_j^*(p)\exp(-ipx)\\&\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),k(\nu-1)+(r+1)(\alpha-\nu(\alpha-\beta))-\nu\delta+j+1}^{r+1}(-\xi t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})\}dp\\&+\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_0^t\{u^{\{\alpha-\nu(\alpha-\beta)\}(r+1)-1}\int_{-\infty}^{\infty}\psi^*(p,t-u)\exp(-ipx)\\&\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),\{\alpha-\nu(\alpha-\beta)\}(r+1)}^{r+1}(-\xi\cdot t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})dp\}du.\end{aligned} \quad (20)$$

Proof. By applying the Sumudu transform on both sides of (18) with respect to variable t, then using (16) and (19), we obtain

$$\begin{aligned}&s^{\nu(\alpha-\beta)-\alpha}\overline{N}(x,s)-\sum_{j=0}^{k-1}s^{\nu(k-\beta)-k+j}f_j(x)+\xi\cdot s^{\nu(\mu-\delta)-\mu}\overline{N}(x,s)\\&-\xi\cdot\sum_{j=0}^{k-1}s^{\nu(k-\delta)-k+j}g_j(x)\;=\chi^2{}_{-\infty}\mathcal{D}_t^{\gamma}\overline{N}(x,s)+\eta^2\overline{N}(x,s)+\overline{\psi}(x,s).\end{aligned} \quad (21)$$

Now applying Fourier transform on both sides of (21) with respect to variable x and using (8), we get

$$s^{\nu(\alpha-\beta)-\alpha}\overline{N}^{*}(p,s)-\sum_{j=0}^{k-1}s^{\nu(k-\beta)-k+j}f_j^*(p)+\xi\cdot s^{\nu(\mu-\delta)-\mu}\overline{N}^{*}(p,s)-\xi\cdot\sum_{j=0}^{k-1}s^{\nu(k-\delta)-k+j}g_j^*(p)$$
$$=-\chi^2|p|^{\gamma}\overline{N}^{*}(p,s)+\eta^2\overline{N}^{*}(p,s)+\overline{\psi}^{*}(p,s).$$

Solving for $\overline{N}^{*}(p,s)$, the above equation is equivalent to

$$\overline{N}^{*}(p,s)=\sum_{j=0}^{k-1}f_j^*(p)\frac{s^{\nu(k-\beta)-k+j}}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}+\rho}$$
$$+\sum_{j=0}^{k-1}\xi\cdot g_j^*(p)\frac{s^{\nu(k-\delta)-k+j}}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}+\rho}+\frac{\overline{\psi}^{*}(p,s)}{s^{\nu(\alpha-\beta)-\alpha}+\xi\cdot s^{\nu(\mu-\delta)-\mu}+\rho}, \tag{22}$$

where $\rho=\chi^2|p|^{\gamma}-\eta^2$. Now by applying inverse Sumudu transform on both sides of (22) and then using (14) and convolution of Sumudu transform (11), we have

$$N^*(p,t)$$
$$=\sum_{j=0}^{k-1}f_j^*(p)\sum_{r=0}^{\infty}\{(-\rho)^r t^{\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k+j}$$
$$\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),\nu(k+r\beta)+\alpha(r+1)(1-\nu)-k+j+1}^{r+1}(-\xi\cdot t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})\}$$
$$+\sum_{j=0}^{k-1}\xi\cdot g_j^*(p)\sum_{r=0}^{\infty}\{(-\rho)^r t^{k(\nu-1)+(r+1)\{\alpha(1-\nu)+\nu\beta\}-\nu\delta+j}$$
$$\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),k(\nu-1)+(r+1)\{\alpha(1-\nu)+\nu\beta\}-\nu\delta+j+1}^{r+1}(-\xi\cdot t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})\}$$
$$+\sum_{r=0}^{\infty}(-\rho)^r\int_0^t\{\psi^*(p,t-u)u^{\{\alpha-\nu(\alpha-\beta)\}(r+1)-1}$$
$$\times E_{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta),\{\alpha-\nu(\alpha-\beta)\}(r+1)}^{r+1}(-\xi\cdot t^{\alpha-\nu(\alpha-\beta)-\mu+\nu(\mu-\delta)})\}du. \tag{23}$$

Thus, applying inverse Fourier transform on both sides of (23), we arrive at (20).

A. Special Cases

Corollary 1 For $\nu=0$, the reaction diffusion equation with GCFD (18) takes the form

$${}_0\mathcal{D}_t^{\alpha}N(x,t)+\xi\cdot{}_0\mathcal{D}_t^{\mu}N(x,t)=\chi^2{}_{-\infty}\mathcal{D}_x^{\gamma}N(x,t)+\eta^2N(x,t)+\psi(x,t), \tag{24}$$

where ${}_0\mathcal{D}_t^{\alpha}$ and ${}_0\mathcal{D}_t^{\mu}$ are fractional derivatives in Riemann-Liouville sense such that $k-1<\alpha,\mu\le k$; $\alpha>\mu$ with the initial conditions

$$\left.\begin{aligned}\mathcal{D}^j{}_0\mathcal{J}_t^{(k-\alpha)}N(x,t)|_{t=0}&=f_j(x)\\ \mathcal{D}^j{}_0\mathcal{J}_t^{(k-\mu)}N(x,t)|_{t=0}&=g_j(x)\end{aligned}\right\};\ j=0,1,2,\dots k-1,\&\ k\in\mathbb{N}. \tag{25}$$

The solution of (24) with the initial conditions (25) is given by

$$N(x,t)=\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{\alpha(r+1)-k+j}f_j^*(p)\exp(-ipx)$$
$$\times E_{\alpha-\mu,\alpha(r+1)-k+j-1}^{r+1}(-\xi\cdot t^{\alpha-\mu})\}dp$$
$$+\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\xi\cdot\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{\alpha(r+1)-k+j}g_j^*(p)\exp(-ipx)$$
$$\times E_{\alpha-\mu,\alpha(r+1)-k+j-1}^{r+1}(-\xi\cdot t^{\alpha-\mu})\}dp$$
$$+\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_0^t\{u^{\alpha(r+1)-1}\int_{-\infty}^{\infty}\psi^*(p,t-u)\exp(-ipx)$$
$$\times E_{\alpha-\mu,\alpha(r+1)}^{r+1}(-\xi\cdot t^{\alpha-\mu})dp\}du.$$

Corollary 2 For $\nu=1$, the reaction diffusion equation with GCFD (18) reduces into

$${}_0\mathcal{D}_t^{\beta}N(x,t)+\xi\cdot{}_0\mathcal{D}_t^{\delta}N(x,t)=\chi^2{}_{-\infty}\mathcal{D}_x^{\gamma}N(x,t)+\eta^2N(x,t)+\psi(x,t), \tag{26}$$

where ${}_0\mathcal{D}_t^{\beta}$ and ${}_0\mathcal{D}_t^{\delta}$ are fractional derivatives in caputo sense such that $k-1<\beta,\delta\le k$; $\beta>\delta$ with the initial conditions

$$\mathcal{D}^jN(x,t)|_{t=0}=\phi_j(x),\ j=0,1,2,\cdots k-1\text{ and }x\in\mathbb{R}. \tag{27}$$

The solution of (26) with the initial conditions (27) is given as

$$N(x,t)=\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\beta r+j}\phi_j^*(p)\exp(-ipx)E_{\beta-\delta,\beta r+j+1}^{r+1}(-\xi t^{\beta-\delta})dp$$
$$+\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\xi\cdot\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{\beta(r+1)-\delta+j}\phi_j^*(p)\exp(-ipx)$$
$$\times E_{\beta-\delta,\beta(r+1)-\delta+j-1}^{r+1}(-\xi\cdot t^{\beta-\delta})\}dp$$
$$+\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_0^t\{u^{\beta(r+1)-1}\int_{-\infty}^{\infty}\psi^*(p,t-u)\exp(-ipx)$$
$$\times E_{\beta-\delta,\beta(r+1)}^{r+1}(-\xi\cdot t^{\beta-\delta})dp\}du. \tag{28}$$

In particular, if we set $k=1$ in (26)-(28), we get a form of fractional reaction diffusion equation studied by [6] and [24].

Corollary 3 For $\alpha=\beta$; $\mu=\delta$, the generalized composite fractional reaction-diffusion (18) reduces into reaction-diffusion equation with Hilfer's fractional derivative, that is

$${}_0\mathcal{D}_t^{\alpha,\nu}N(x,t)+\xi\cdot{}_0\mathcal{D}_t^{\mu,\nu}N(x,t)=\chi^2{}_{-\infty}\mathcal{D}_x^{\gamma}N(x,t)+\eta^2N(x,t)+\psi(x,t), \tag{29}$$

with the corresponding initial conditions as:

$$\left.\begin{array}{rcl}\mathcal{D}^j{}_0\mathcal{J}_t^{(1-\nu)(k-\alpha)}N(x,t)|_{t=0} & = & f_j(x)\\ \mathcal{D}^j{}_0\mathcal{J}_t^{(1-\nu)(k-\mu)}N(x,t)|_{t=0} & = & g_j(x)\end{array}\right\};\ j=0,1,2,\dots k-1,\&\ k\in\mathbb{N}, \tag{30}$$

where $k-1<\alpha,\mu\le k$ such that $\alpha>\mu$, $0\le\nu\le1$. The corresponding solution is given by

$$N(x,t)=\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{\alpha(r+1-\nu)+\nu(k-1)+j}f_j^*(p)\exp(-ipx)$$
$$\times E_{\alpha-\mu,\alpha(r+1-\nu)+k(\nu-1)+j+1}^{r+1}(-\xi\cdot t^{\alpha-\mu})\}dp$$
$$+\sum_{j=0}^{k-1}\sum_{r=0}^{\infty}\xi\cdot\frac{(-\rho)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{t^{k(\nu-1)+\alpha(r+1)-\nu\mu+j}g_j^*(p)\exp(-ipx)$$
$$\times E_{\alpha-\mu,k(\nu-1)+\alpha(r+1)-\nu\mu+j+1}^{r+1}(-\xi\cdot t^{\alpha-\mu})\}dp$$
$$+\sum_{r=0}^{\infty}\frac{(-\rho)^r}{\sqrt{2\pi}}\int_0^t\{u^{\alpha(r+1)-1}\int_{-\infty}^{\infty}\psi^*(p,t-u)\exp(-ipx)$$
$$\times E_{\alpha-\mu,\alpha(r+1)}^{r+1}(-\xi\cdot t^{\alpha-\mu})\}dudp.$$

In particular, if we take $k=1$, then this case gives fractional diffusion equation studied by [7].

IV. Conclusion

In this work, first we have given some lemmas related to Sumudu transform and then the Sumudu transform of generalized composite fractional derivative. We have generalized the fractional reaction diffusion equation by using Generalized Composite Fractional Derivative (GCFD) and we obtained the solution of reaction diffusion equation with GCFD by the use of Sumudu and Fourier transform. Further, we have mentioned some special cases related to the generalized equation.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

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