Comparison of Laplace Beltrami Operator Eigenvalues on Riemannian Manifolds

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***Abstract* — Let Δ_g be the Laplace Beltrami operator on a manifold M with Dirichlet (resp., Neumann) boundary conditions. We compare the spectrum of on a Riemannian manifold for Neumann boundary condition and Dirichlet boundary condition. Then we construct an effective method of obtaining small eigenvalues for Neumann's problem.**

***Keywords* — Eigenvalue problem, Laplacian, manifold, spectrum.**

I. INTRODUCTION

In daily life, physical phenomena such as heat diffusion, wave propagation, quantum mechanics are described by Laplacian operators. In this paper, to make things more generic, we will stick to the level of connected Riemannian manifolds (M, g) associated with Riemannian metric g .Throughout this paper we assume that the manifold M with boundary ∂M .The Laplacian we will consider is given by the form:

$$\Delta_g = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(g^{ij} \sqrt{|\det g|} \frac{\partial}{\partial x_i} \right). \tag{1}$$

Δ_g is called Laplace Beltrami operator where $(x_1, ..., x_n)$ are local coordinates in M .

From now on, we examine the spectrum of $-\Delta_g$ on M , we deal the eigenvalue problem:

$$-\Delta_g \varphi = \lambda \varphi. \tag{2}$$

For $\varphi \in C^2(M)$.

Under the Dirichlet condition $\varphi|_{\partial M} = 0$,the spectrum consists of real and discrete eigenvalues:

$$0 < \lambda_1(M) \le \lambda_2(M) \le .... \to \infty. \tag{3}$$

The Neumann boundary condition i.e., $\left.\left(\frac{\partial \varphi}{\partial n}\right)\right|_{\partial M} = 0$ where n denotes the exterior normal to the boundary.

The spectrum $\sigma(-\Delta_g)$ consists of discrete and real sequence:

$$0 = \lambda_1(M) \le \lambda_2(M) \le .... \to \infty. \tag{4}$$

With the eigenvalues repeated according to their multiplicity. See reference [1] for the proof.

Notation. Sometimes, we will write $\lambda_k^D(M)$ and $\lambda_k^N(M)$ to avoid confusion between the eigenvalues with respect to the Dirichlet boundary conditions and Neumann boundary conditions.

Next, the minimax characterization is presented in reference [2], which is a fundamental tool for examining the spectrum using geometry.

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The minimax characterization is given by the formula:

$$\lambda_k = \inf_{V_k} \sup\left\{R(\varphi) : \varphi \neq 0, \varphi \in V_k\right\}. \tag{5}$$

Where $R(\varphi)$ is the Rayleigh quotient of φ:

$$R(\varphi) = \frac{\int_M |\nabla\varphi|^2 \, dV_g}{\int_M \varphi^2 dV_g} = \frac{(\nabla\varphi, \nabla\varphi)}{(\varphi, \varphi)}. \tag{6}$$

and V_k runs through k- dimensional subspaces of $H_0^1(M)$ for Dirichlet eigenvalue problem and the $k+1$ dimensional subspaces of $H^1(M)$ for the Neumann eigenvalue problem.

I would like to mention a very interesting book about Riemannian manifolds by Petersen, Peter in reference [3]. Also interested investigators in spectral theory can look at reference [4].

In this paper, the first section presents properties of the Laplace-Beltrami spectral theorem. For the problem of hearing the length of a guitar string, the spectral interpretation is presented in the second section. The third section compares the Dirichlet and Neumann problems.

II. Properties in Spectral Theorem for the Laplace–Beltrami

Let (M, g) be a compact Riemannian manifold with boundary ∂M . Reference [5] shows that by using Green's formula:

$$\int_M \psi \cdot \Delta_g \varphi \, dV_g = -\int_M \langle \nabla\psi, \nabla\varphi \rangle_g \, dV_g + \int_{\partial M} \varphi \partial_\nu \varphi \, d\sigma_g. \tag{7}$$

For all $\psi \in C^1(M)$ and $\varphi \in C^2(M)$,we conclude that the relation for the case of manifold without boundary:

$$\int_M \varphi \Delta_g \psi dV_g = \int_M \Delta_g \varphi \psi dV_g. \tag{8}$$

Hence, the operator $-\Delta_g$ is symmetric. We have also:

$$-\int_M \varphi \Delta_g \varphi \, dV_g = \int_M |\nabla\varphi|^2 dV_g \geq 0. \tag{9}$$

therefore the operator $-\Delta_g$ is positive.

Consequently, let λ, μ be two eigenvalues such that $\lambda \neq \mu$ and let u, v be respective eigenfunctions (i.e., $-\Delta_g u = \lambda u$ and $-\Delta_g v = \mu v$. Since $-\Delta_g$ is symmetric, we have:

$$\langle -\Delta_g u, v \rangle_{L^2} = \langle u, -\Delta_g v \rangle_{L^2}$$

i.e., $(\lambda - \mu)\langle u, v \rangle_{L^2} = 0$.

Therefore, if we denote by $E(\lambda)$ the eigenspace corresponding to the eigenvalue λ , the spaces $E(\lambda)$ and $E(\mu)$ are L^2 -orthogonal. Moreover, every eigenvalue is non-negative. Indeed, let λ be an eigenvalue and let u be an eigenfunction of λ . Then:

$$\langle -\Delta_g u, u \rangle_{L^2} = \lambda \|u\|_{L^2} \geq 0.$$

so $\lambda \geq 0$.

Now ,the minimax formula doesn't help calculating λ_k, but it is very useful to find an upper bound for instance , in the case of the Neumann (or Dirichlet) problem, for any given (k + 1) dimensional of $H_0^1(M)$ (or. k dimensional) vector subspace V of $H^1(M)$, that is:

$$\lambda_k(M) \le \sup\{R(\varphi) : \varphi \neq 0, \varphi \in V\}. \tag{10}$$

This gives immediately an upper bound for $\lambda_k(M)$ if it is possible to estimate the Rayleigh quotient $R(\varphi)$ of all the functions $\varphi \in V$ .Note that there is no need to calculate the Rayleigh quotient, it suffices to estimate it from above.

For reviewing some results concerning in spectral theory of the Laplacian on Riemannian manifolds see references [6]-[9].

Proposition. In the case of Neumann boundary conditions the first eigenvalue is zero i.e., $\lambda_1 = 0$ and the constant $M \ni x \mapsto 1$ is the eigenfunction corresponding to λ_1 .In the case of the Dirichlet boundary condition the first eigenvalue satisfies $\lambda_1 > 0$.

Proof. By the spectral theorem, $\lambda_1 \ge 0$, and minimax principle we also get:

$$\lambda_1 = \inf_{V_k} \frac{\int_M |\nabla \varphi|^2 \, dV_g}{\int_M \varphi^2 dV_g}.$$

Since the constant function $x \mapsto 1$ belongs to the space $H^1(M)$ and

$$\frac{\int_M |\nabla 1|^2 \, dV_g}{\int_M 1^2 dV_g} = 0.$$

We conclude that $\lambda_1 \le 0$,therefore $\lambda_1 = 0$. Moreover, $-\Delta_g(1) = 0$.

For the Dirichlet problem and by contradiction assume that $\lambda_1 = 0$,there exists a non-trivial function u such that $-\Delta_g(u) = 0$ and u vanishes on the boundary of M .Therefore,

$$\int_M -\Delta_g u \, u dV_g = 0.$$

Hence integrating by parts, we get:

$$\int_M |\nabla u|^2 \, dV_g = 0.$$

Thus u is constant on M and from the boundary conditions it follows that $u = 0$ on M , which is a contradiction.

III. Hearing the Length of Guitar String with Dirichlet Boundary Condition

Consider M is an interval $[0, L]$ with a Dirichlet boundary condition. This example is a model of vibration of a fixed string in dimension one (this subsection is described as of direct problem in spectral geometry see reference [10]). The spectrum depends only on the length of string, mass, and tension. In this paper we assume that the density mass and tension are constant 1.

For notational simplicity we write Δ instead of Δ_g for Euclidean domains.

Now, let $\varphi \in C^2([0, L])$,where $\varphi(0) = \varphi(L) = 0$.

The eigenfunctions are $\sin\left(\frac{k\pi}{L}x\right)$ for $k \ge 1$ with eigenvalues $\lambda_k = \left(\frac{k\pi}{L}\right)^2$.

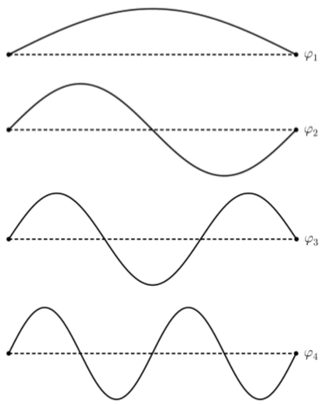

Fig. 1. Waves of guitar string.

The way in which string vibrates is described by wave equation, meaning that ,if x any point in string $[0,L]$ at the time t , the height $u(x,t)$ satisfies:

$$\Delta u(x,t)=\frac{\partial^2}{\partial t^2}u(x,t).$$

The expression of solution is:

$$u(x,t)=\sum_{k=1}^{\infty}\alpha_k(t)\varphi_k(x).$$

where

$$\alpha_k=a_k\cos\left(\frac{k\pi}{L}t\right)+b_k\sin\left(\frac{k\pi}{L}t\right).$$

The coefficients are $a_k=\langle f,\varphi_k\rangle$, $b_k=\langle g,\varphi_k\rangle$ for the functions $f(x)=u(x,0)$, $g(x)=\partial u_t(x,0)$.
Consider the Fourier transform of φ :

$$F(\varphi)(\zeta)=\int_{-\infty}^{\infty}\varphi(x)e^{-ix\zeta}.$$

Then for $\varphi_k(x)=\sin\left(\sqrt{\lambda_k}x\right)$,we have:

$$F(\varphi_k(\zeta))=\frac{\pi}{i}\left(\delta\left(\zeta-\sqrt{\lambda_k}\right)-\delta\left(\zeta+\sqrt{\lambda_k}\right)\right).$$

Since, $\sin\left(\sqrt{\lambda_k}x\right)=\frac{e^{i\sqrt{\lambda_k}x}-e^{i\sqrt{\lambda_k}x}}{2i}$ and $F\left(e^{i\sqrt{\lambda_k}x}\right)(\zeta)=2\pi\delta\left(\zeta-\sqrt{\lambda_k}\right)$.
If you pluck a guitar string, then you get a wave of the form:

$$F(u(.,t)\zeta)=u(x,t)=\frac{\pi}{i}\sum_{k=1}^{\infty}\alpha_k(t)\left(\delta\left(\zeta-\sqrt{\lambda_k}\right)-\delta\left(\zeta+\sqrt{\lambda_k}\right)\right).$$

Remark. For Neumann boundary condition, Laplacian is given by $\Delta\varphi=-\varphi''$ and is the spectrum is given by $\lambda_k=\frac{k^2\pi^2}{L^2}$, k = 0,1,2,.. The eigenfunction corresponding to λ_k is $\varphi_k(x)=\cos\frac{k\pi}{L}x$.

IV. Comparison between Dirichlet and Neumann Problem

Let (M,g) be a Riemannian manifold and A, B two submanifolds of same dimension n. If $A \subset B$,then we have that $\lambda_k^D(A) \geq \lambda_k^D(B)$ for every integer $k \geq 1$ .For the Dirichlet problem and by using the minimax characterization of the spectrum,we have: each eigenfunction of A can be a continuously extended by 0 on B and may be used as a test function for the Dirichlet problem on B. Let us construct an upper bound for $\lambda_k^D(B)$: for V_k, we choose the vector subspace of $H_0^1(B)$ generated by an orthonormal basis $\varphi_1,...,\varphi_k$ of eigenfunctions of A extended by 0 on B. Clearly, these functions vanish on ∂B and they are C^∞ on A and $B \cap \overline{A}^c$. They are continuous on ∂A .

Let $\varphi = \alpha_1\varphi_1 + ... + \alpha_n\varphi_n$. We have:

$$(\varphi,\varphi) = \alpha_1^2 + \alpha_2^2 + ... + \alpha_k^2.$$

Then

$$\begin{aligned}(\nabla\varphi_i,\nabla\varphi_j) &= \int_B \langle \nabla\varphi_i,\nabla\varphi_j\rangle dV_g = \int_A \langle \nabla\varphi_i,\nabla\varphi_j\rangle dV_g \\ &= \int_A \langle \Delta_g\varphi_i,\varphi_j\rangle dV_g = \lambda_i\int_A \varphi_i\varphi_j dV_g,\end{aligned}$$

It follows that $(\nabla\varphi_i,\nabla\varphi_i) = \lambda_i^D(A)(\varphi_i,\varphi_i)$ and $(\nabla\varphi_i,\nabla\varphi_j) = 0$ if $i \neq j$ .We have $(\nabla\varphi,\nabla\varphi) = \alpha_1^2\lambda_1^D(A) + ... + \alpha_k^2\lambda_k^D(A) \leq \lambda_k^D(A)(\alpha_1^2 + ... + \alpha_k^2)$. We conclude that $R(\varphi) \leq \lambda_k^D(A)$, and we have $\lambda_k^D(B) \leq \lambda_k^D(A)$. We realize from the proof that a test function for the Dirichlet problem is also a test function for the Neumann problem,

$$\lambda_k^N(B) \leq \lambda_{k+1}^D(A).$$

and, in particular, taking $A = B = M$,we have:

$$\lambda_k^N(M) \leq \lambda_{k+1}^D(M).$$

We can construct $A \subset B$ with as many eigenvalues as small as follow:

Motivated by construction of small eigenvalues for the Neumann problem we use the Cheeger dumbbell construction:

Consider two n-balls of fixed volume A connected by a small cylinder C of length $2L$ and radius ε .We denote by Ω_ε this domain. The first nonzero eigenvalue of Ω_ε converges to 0 as ε tends to 0. Let us shows that λ_1 converges to 0.

We define the function f on the first ball with value 1, f takes the value -1 on the second ball and decreasing linearly along the cylinder.

The maximum norm of its gradient is $1/\mathrm{L}$.

By construction (and for simplicity we assume that the manifold is symmetric), we have $\int_M f dV_g = 0.$

We introduce the vector space V generated by f and by the constant 1.

For $h \in V$,we can write $h = a + bf, a,b \in \mathbb{R}$ and

$$\int_{\Omega_\varepsilon} h^2 dx = a^2 Vol(\Omega_\varepsilon) + b^2 \int_{\Omega_\varepsilon} f^2 dx,$$

$$\int_{\Omega_\varepsilon} |\nabla h|^2 dx = b^2 \int_{\Omega_\varepsilon} |\nabla f|^2 dx.$$

By the minimax characterization, we have $\lambda_1 \leq \sup\{R(h) : \mathrm{h} \neq 0, \mathrm{h} \in V\}$ and we get $\lambda_1(\Omega_\varepsilon) \leq R(f)$.

The function f varies only on the cylinder C and its gradient has norm $1/\mathrm{L}$ .This implies

$$\int_{M_\varepsilon} |df|^2 \, dVol_g = 1/L^2\left(Vol\left(C\right)\right).$$

Moreover, because f^2 takes the value 1 on both balls of volume A, we have:

$$\int_{\Omega_\varepsilon} f^2 dx \geq 2A.$$

This implies that the Rayleigh quotient of f is bounded above by $\frac{VolC / L^2}{2A}$ which tends to 0 as ε does.

We want to focus on the Neumann problem, since this case is completely different. We will show by example that we have no monotonicity for the Neumann problem.

Example. Let M be a Euclidean domain. There exists a domain $A \subset M$ with a smooth boundary such that the k first eigenvalues of A for the Neumann problem are arbitrarily small. To do this choose A a Cheeger dumbbell, with k+1 balls joined by very thin cylinders. We see immediately that the k first eigenvalues can be made as small as we wish, with the same calculations as above.

V. CONCLUSION

We analyzed the monotony property of the Neumann problem and the Dirichlet problem. we interpreted geometrical consequences of spectrum of the length of a guitar string.

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CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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