F_g-Metric Spaces

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***Abstract* — This paper introduces the concept of F_g-metric Space. We generalize the concept of G-metric space. Some supporting examples are given.**

***Keywords* — F_g-metric space, F_g-metric, Generalized metric space.**

I. Introduction

2-metric spaces as a generalization of metric spaces were introduced by Gahler [1]-[2]. The definition is as follows,

Definition 1.1. Let X be a nonempty set. A function $d: X \times X \times X \to R$ satisfying the following properties:

(A1) For distinct points $x, y \in X$, there is $z \in X$, such that $d(x, y, z) \neq 0$,

(A2) $d(x, y, z) = 0$ if two of the triple $x, y, z \in X$ are equal,

(A3) $d(x, y, z) = d(x, z, y) = d(y, z, x) = \cdots$, (symmetry in all three variables),

(A4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality),

is called 2-metric, on X. The set X equipped with such 2-metric is called a 2-metric space. Some authors disproved Gahler's claim that a 2-metric is a generalization of the usual notion of a metric. They proved that these spaces have no relation [3]. Bapurao Dhage tried to rectify these flaws and introduced a new class of generalized metrics called D-metrics [4]-[8].

Definition 1.2. A function $D: X \times X \times X \to R$ is a D-metric if it satisfies axioms (A3) and (A4), but with (A1) and $(A2)$ replaced by the single axiom:

(A0) $D(x, y, z) = 0$ if and only if $x = y = z$.

An additional property sometimes imposed by Dhage on a D-metric is,

(A5) $D(x, y, y) \leq D(x, z, z) + D(z, y, y)$ for all $x, y, z \in X$.

Dhage claimed that metric functions are special case of D-metrics and proved many fixed point results in D-metric spaces as a generalization of such results in metric spaces. But Zead Mustafa and Brailey Sims pointed some flaws in claims of Dhage regarding fundamental topological properties of D-metric spaces, D-convergence of a sequence (x_n) to x and continuity of D metric function in its variables[9]. Keeping these flaws in mind Zead Mustafa and Brailey Sims came up with more appropriate notion of generalized metric space.

Definition 1.3. Let X be a nonempty set and let $G: X \times X \times X \to R$, be a function satisfying the following:

(G1) $g(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < g(x, x, y)$; for all $x, y \in X$, with $x \neq y$,

(G3) $g(x, x, y) \leq g(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G4) $g(x, y, z) = g(x, z, y) = g(y, z, x) = \cdots$, (symmetry in all three variables),

(G5) $g(x, y, z) \leq g(x, a, a) + g(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)

In 2018 M. Jleli and B. Samet coined the concept of F-metric space as generalization of the metric space and they did comparative study of F-metric space with existing generalization ofmetric spaces. Also, they discussed natural topology on these spaces and proved a new version of the Banach contraction principle in the setting of F-metric spaces [10].

II. Preliminaries

Reference [11] considered a nonlinear function $F: (0, \infty) \to R$ with the following characteristics:

(F1) F is strictly increasing,

(F2) for any sequence $\{t_n\} \subset (0, \infty)$, we have

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$$\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} F(t_n) = -\infty,$$

(F3) there exists $l \in (0,1)$ such that $\lim_{t\to 0^+} t^l F(t) = 0$.

Let $B = \{F: (0,\infty) \to R / F$ satisfies $(F_1) - (F_2)\}$.

Definition 2.1. Let X be a nonempty set. Suppose that there exists $(f, \alpha) \in B \times [0, \infty)$ and let $g : X \times X \times X \to R$, be a function satisfying the following:

(g1) $g(x, y, z) = 0$ if $x = y = z$,

(g2) $0 < g(x, x, y)$; for all $x, y \in X$, with $x \neq y$,

(g3) $g(x, x, y) \leq g(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(g4) $g(x, y, z) = g(x, z, y) = g(y, z, x) = \cdots$, (symmetry in all three variables),

(g5) for every $x, y, z \in X$ and $(u_i)_{i=1}^m \subset X, m \in N, m \geq 2$, with $u_1 = x$, we have

(g6)

$$g(x, y, z) > 0 \text{ implies } f(g(x, y, z)) \leq f\left[\left(\sum_{i=1}^{m-1} \left(g(u_i, u_{i+1}, u_{i+1})\right) + g(u_m, y, z)\right]\right. + \alpha.$$

Then g is said to be F_g-metric and the pair (X, g) is said to be F_g-metric space.

It is important to note that $g(x, x, y)$ and $g(x, y, y)$ need not be equal in F_g-metric space. Consider the following example.

Example 2.2. Let $X = \{a, b\}$ and define g on X as $g(a, a, a) = g(b, b, b) = 0, g(a, a, b) = g(a, b, a) = g(b, a, a) = 2$ and $g(a, b, b) = g(b, a, b) = g(b, b, a) = 1$. It can be easily verified that with $f = \log$ and $\alpha = 0, (X, g)$ forms F_g-metric space, where $g(a, a, b) \neq g(a, b, b)$.

Definition 2.3. A F_g-metric space (X, g) is said to be symmetric if

$$g(x, x, y) = g(x, y, y) \text{ for all } x, y \in X.$$

We can see that F_g-metric space is generalization of G-metric space. To prove this, first we will prove that every G-metric space is F_g-metric space.

If g is a G-metric spaces then (g1) to (g4) follows directly. Also, from $(G5)$ of G-metric space we have, for $u_1 = x, m \geq 2, m \in N$ and $(u_i)_{i=1}^m, y, z \in X$

$$g(x, y, z) \leq \left[\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right] + g(u_m, y, z).$$

If $g(x, y, z) > 0$, then

$$\log\left(g(x, y, z)\right) \leq \log\left(\left[\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right] + g(u_m, y, z)\right).$$

So g satisfies $(g5)$ with $f = \log$ and $\alpha = 0$. Now we present examples of F_g-metric spaces which are not G-metric spaces.

Example 2.4. Let $X = N$. Let us define g as

$$g(x, y, z) = \begin{cases} 0 & : \text{if } x = y = z \\ \exp\left(|x - y| + |y - z| + |z - x|\right) & : \text{otherwise} \end{cases}$$

Here we can observe that $g(1,2,4) = \exp(6), g(1,3,3) = \exp(4)$ and $g(3,2,4) = \exp(4)$. So clearly g is not $G -$ metric as it doesn't satisfy (G5),

$$g(1,2,4) = \exp(6) > 2exp(4) = g(1,3,3) + g(3,2,4).$$

Now we will prove that g is F_g-metric space. It can be easily verified that g satisfies $(g1)$, (g2) and $(g4)$. We will just verify $(g3)$ and $(g5)$. By the definition of g, we have

$$g(x, y, y) = \exp\left(|x - y| + |y - x|\right)$$
$$g(x, y, z) = \exp\left(|x - y| + |y - z| + |z - x|\right).$$

Also, we know that

$$\begin{aligned} |x-y| &\le |x-z|+|z-y| \\ |x-y|+|x-y| &\le |x-y|+|x-z|+|z-y| \\ \exp(|x-y|+|x-y|) &\le \exp(|x-y|+|x-z|+|z-y|) \\ g(x,y,y) &\le g(x,y,z). \end{aligned}$$

Thus (g3) is satisfied here. Now we will prove $(g5)$. Let us fix certain $x, y, z \in X$ with $g(x, y, z) > 0$. Let $(u_i)_{i=1}^{m} \subset X$, where $m \in N, m \ge 2$, and $u_1 = x$. Let in the range of $i = 1$ to $i = m - 1$ sum runs over index set I if all three points are equal and sum runs over index set J otherwise.

$$\begin{aligned} & 1+f\left(\sum_{i=1}^{m-1}\left(g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right)-f(g(x,y,z)) \\ = \quad & 1-\frac{1}{\sum_{i=1}^{m-1}\left(g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)}+\frac{1}{g(x,y,z)} \\ = \quad & 1-\frac{1}{\sum_J\left(|u_i-u_{i+1}|+|u_{i+1}-u_{i+1}|+|u_{i+1}-u_i|\right)+g(u_m,y,z)}+\frac{1}{\exp(|x-y|+|x-z|+|z-y|)}. \end{aligned}$$

Also, we have $g(u_m, y, z)$ has value 0 if $u_m = y = z$ otherwise it has valueexp $(|u_m - y| + |y - z|+| z - u_m |)$. So, in any case we have

$$\frac{1}{\sum_J\left(|u_i-u_{i+1}|+|u_{i+1}-u_{i+1}|+|u_{i+1}-u_i|\right)+g(u_m,y,z)} \le 1.$$

Therefore

$$\begin{aligned} & 1+f\left(\sum_{i=1}^{m-1}\left(g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right)-f(g(x) \\ & \ge 1-1+\frac{1}{\exp(|x-y|+|x-z|+|z-y|)} \\ & =\frac{1}{\exp(|x-y|+|x-z|+|z-y|)} \\ & \ge 0. \end{aligned}$$

Hence

$$f(g(x,y,z)) \le f\left(\sum_{i=1}^{m-1}\left(g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right)+1.$$

That is, (g_5) is satisfied and (X, g) is F_g-metric space with $f = \frac{-1}{t}$ and $\alpha = 1$.

Example 2.5. Let $X =$ Set of nonnegative integers. Let us define $g: X \to R$ as

$$g(x,y,z)=\begin{cases}\max\{(x-y)^2,(y-z)^2,(z-x)^2\} & : \text{if } x,y,z\in\{0,1,2,3\} \\ \max\{|x-y|,|y-z|,|z-x|\} & : \text{otherwise.}\end{cases}$$

Here we can observe that $g(0,1,3) = 9, g(0,2,2) = 4$ and $g(2,1,3) = 4$. So clearly g is not G metric. Since it doesn't satisfy (G5), as

$$g(0,1,3)=9>4+4=g(0,2,2)+g(2,1,3).$$

Now we will prove that g is F_g-metric space. It can be easily verified that g satisfies $(g1)$, (g2) and $(g4)$. We will just verify $(g3)$ and (g5). By the definition of g, we have

$$g(x,x,y)=\begin{cases}(x-y)^2 & \text{: if } x,y,z\in\{0,1,2,3\}\\ |x-y| & \text{: otherwise.}\end{cases}$$
$$g(x,y,z)=\begin{cases}\max\{(x-y)^2,(y-z)^2,(z-x)^2\} & \text{: if } x,y,z\in\{0,1,2,3\}\\ \max\{|x-y|,|y-z|,|z-x|\} & \text{: otherwise.}\end{cases}$$

Since $|x-y|\le\max\{|x-y|,|y-z|,|z-x|\}$ and $(x-y)^2\le\max\{(x-y)^2,(y-z)^2,(z-x)^2\}$, we have

$$g(x,x,y)\le g(x,y,z),\text{ for all } x,y,z\in X \text{ with } z\ne y.$$

Let us fix certain $x,y,z\in X$ with $g(x,y,z)>0$. Let $(u_i)_{i=1}^m\subset X$, where $m\in N, m\ge 2$ and $u_1=x$. Let in the range of $i=1$ to $i=m-1$ sum runs over index set I if all three points lies in the set $\{0,1,2,3\}$ and sum runs over index set J otherwise.

$$\begin{aligned}\sum_{i=1}^{m-1} g(u_i,u_{i+1},u_{i+1}) &= \sum_{i\in I} g(u_i,u_{i+1},u_{i+1})+\sum_{i\in J} g(u_i,u_{i+1},u_{i+1})\\ &= \sum_{i\in I}\max\{(u_i-u_{i+1})^2,(u_{i+1}-u_{i+1})^2,(u_{i+1}-u_i)^2\}\\ &\quad+\sum_{i\in J}\max\{|u_i-u_{i+1}|,|u_{i+1}-u_{i+1}|,|u_{i+}-u_i|\}.\end{aligned}$$

Consider the following two cases,
Case I: If $(x,y,z)\notin\{0,1,2,3\}$.

$$\begin{aligned}g(x,y,z) &= \max\{|x-y|+|y-z|+|z-x|\}\\ &\le \sum_{i=1}^{m-1}\max\{|u_i-u_{i+1}|+|u_{i+1}-u_{i+1}|+|u_{i+1}-u_i|\}\\ &\quad+\max\{|u_m-y|+|y-z|+|z-u_m|\}\\ &\le \sum_{i\in I}\max\{|u_i-u_{i+1}|,|u_{i+1}-u_{i+1}|,|u_{i+1}-u_i|\}\\ &\quad+\sum_{i\in J}\max\{|u_i-u_{i+1}|,|u_{i+1}-u_{i+1}|,|u_{i+1}-u_i|\}\\ &\quad+\max\{|u_m-y|,|y-z|,|z-u_m|\}.\end{aligned}$$

Using inequality, $|x-y|\le(x-y)^2$ for all $x,y\in Z$, we have

$$\begin{aligned}g(x,y,z)\le &\sum_{i\in I}\max\{(u_i-u_{i+1})^2,(u_{i+1}-u_{i+1})^2,(u_{i+1}-u_i)^2\}\\ &+\sum_{i\in J}\max\{|u_i-u_{i+1}|,|u_{i+1}-u_{i+1}|,|u_{i+1}-u_i|\}\\ &+\max\{|u_m-y|,|y-z|,|z-u_m|\},\text{ if } u_m,y,z\notin\{0,1,2,3\}.\end{aligned}$$

And

$$\begin{aligned}g(x,y,z)\le &\sum_{i\in I}\max\{(u_i-u_{i+1})^2,(u_{i+1}-u_{i+1})^2,(u_{i+1}-u_i)^2\}\\ &+\sum_{i\in J}\max\{|u_i-u_{i+1}|,|u_{i+1}-u_{i+1}|,|u_{i+1}-u_i|\}\\ &+\max\{(u_m-y)^2,(y-z)^2,(z-u_m)^2\},\text{ if } u_m,y,z\in\{0,1,2,3\}.\end{aligned}$$

Therefore

$$g(x,y,z)\le\sum_{i=1}^{m-1}\left(g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z).$$

Case II: If $x, y, z \in \{0,1,2,3\}$.

$$\begin{aligned} g(x,y,z) &= \max\{(x-y)^2,(y-z)^2,(z-x)^2\} \\ &\leq 3\max\{|x-y|,|y-z|,|z-x|\}. \end{aligned}$$

Following same steps as in Case I, we get,

$$g(x,y,z) \leq 3\left(\sum_{i=1}^{m-1} \big(g(u_i,u_{i+1},u_{i+1})\big) + g(u_m,y,z)\right).$$

So, in any case, for every$x, y, z \in X$, $(u_i)_{i=1}^m \subset X$, where $m \in N, m \geq 2$ and $u_1 = x$, we have,

$$g(x,y,z) \leq 3\left(\sum_{i=1}^{m-1} \big(g(u_i,u_{i+1},u_{i+1})\big) + g(u_m,y,z)\right).$$

If $g(x,y,z) > 0$, then

$$\begin{aligned} \log\ g(x,y,z) &\leq \log\left[3\left(\sum_{i=1}^{m-1} \big(g(u_i,u_{i+1},u_{i+1})\big) + g(u_m,y,z)\right)\right] \\ &= \log\ 3 + \log\left(\sum_{i=1}^{m-1} \big(g(u_i,u_{i+1},u_{i+1})\big) + g(u_m,y,z)\right). \end{aligned}$$

This proves that g satisfies (g_5) with $f(t) = \log\ t, t > 0$ and $\alpha = \log\ 3$. Thus g is an F_g- metric on X.

III. THE F_g-METRIC TOPOLOGY

Definition 3.1. Let (X, g) be an F_g-metric space and $\{x_n\}$ be sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if for every $\epsilon > 0$ there exists $N \in N$, such that $g(x, x_n, x_n) < \epsilon$ or $g(x, x, x_n) < \epsilon$ for all $n \geq N$. We say that the sequence $\{x_n\}$ is F_g-convergent to x.

Definition 3.2. Let (X, g) be an F_g-metric space. Then the sequence $\{x_n\}$ is said to be F_g-Cauchy if for every $\epsilon > 0$, there exists $N \in N$, such that $g(x_l, x_m, x_n) < \epsilon$ for all $n, m, l \geq N$.

Definition 3.3. A F_g-metric space (X, g) is said to be F_g-complete or complete F_g-metric space if every F_g-Cauchy sequence in (X, g) is F_g-convergent in (X, g).

Definition 3.4. Let (X, g) be an F_g-metric space. Then for $x_0 \in X, r > 0, F_g$-open ball $B(x_0, r)$ and F_g-close ball $B(x_0, r)$ are defined as

$$\begin{aligned} B(x_0,r) &= \{y \in X: g(x_0,y,y) < r\}, \\ B(x_0,r) &= \{y \in X: g(x_0,y,y) \leq r\}. \end{aligned}$$

Definition 3.5. Let (X, g) be an F_g-metric space.

i. A set $O \subset X$ is said to be F_g-open in X, iffor every $x \in O$ there exist $r > 0$ such that $B(x_0, r) \subset O$.

ii. A set $C \subset X$ is said to be F_g-close in X if $X \setminus C$ is F_g-open in X.

Proposition 3.6. Let (X, g) be an F_g-metric space. Then collection of all F_g-open sets in X forms topology, that is natural topology generated by F_g-metric g on X.

Proposition 3.7. Let (X, g) be an F_g-metric space. A set $A \subset X$ is F_g-close in X if and only if for every sequence $\{x_n\}$ in A,

$$\lim_{n\to\infty} g(x, x_n, x_n) = 0, x \in X \text{ implies } x \in A. \tag{1}$$

Proof. Let A is F_g-closed and $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} g(x, x_n, x_n) = 0, x \in X$. Suppose, if possible, $x \in X \setminus A$. But $X \setminus A$ is F_g-open. So there exist some $r > 0$ such that $B(x,r) \subset X \setminus A$. That is, $B(x,r) \cap A = \phi$. Moreover, $\lim_{n\to\infty} g(x, x_n, x_n) = 0$ implies, for $r > 0$ there exist some $N \in N$ such that $g(x, x_n, x_n) < r$, for all $n \geq N$. That is, $x_n \in B(x,r)$ for all $n \geq N$. So $x_n \in B(x,r) \cap A$. This is a contradiction. Hence our supposition $x \in X \setminus A$ is wrong and $x \in A$.

Now we prove converse part. Let "(1)" is satisfied. To prove A is F_g-closed, it is equivalent to prove $X \setminus A$ is F_g-open. Let $y \in X \setminus A$. Suppose, if possible, for each $r > 0$, there exist $x_r \in A$ such that $x_r \in B(y,r) \cap A$. So, for $n \in N$, we have $x_n \in B\left(y, \frac{1}{n}\right) \cap A$. That is, $\{x_n\}$ is a sequence in $X \setminus A$ such that $\lim_{n\to\infty} g(y, x_n, x_n) = 0$. By "(1)" we have $y \in A$. Which is contradiction to the fact that $y \in X \setminus A$. Hence our supposition, for each $r > 0$, there exist $x_r \in A$ such that $x_r \in B(y,r) \cap A$, is wrong. Which proves that there exist some F_g-open ball containing y and contained in $y \in X \setminus A$. That is, $X \setminus A$ is F_g-open and consequently A is F_g-closed.

Proposition 3.8. Let (X, g) be an F_g-metric space. If for any sequence $\{x_n\}$ in X,

$$\lim_{n\to\infty} g(x, x_n, x_n) = 0, x \in X \text{ implies } g(y, x, x) \leq \limsup_{n\to\infty} g(y, x_n, x_n), y \in X. \tag{2}$$

Then $B(a,r)$ is F_g-close in X.

Proof. Let $\{x_n\}$ be a sequence in $B(a,r)$ and $\lim_{n\to\infty} g(x, x_n, x_n) = 0$. In the light of proposition (7), to prove that $B(a,r)$ is F_g-close in X, it is sufficient to prove that $x \in B(a,r)$. As $\{x_n\} \subset B(a,r)$, we have $g(a, x_n, x_n) \leq r$, for all $n \in N$. Which implies,

$$\limsup_{n\to\infty} g(a, x_n, x_n) \leq r$$

Using condition "(2)", we have

$$g(a, x, x) \leq \limsup_{n\to\infty} g(a, x_n, x_n) \leq r.$$

It implies $g(a, x, x) \leq r$. That is, $x \in B(a,r)$. Which proves $B(a,r)$ is F_g-closed in X.

Example 3.9. Consider the function $h: R_2 \to [0, \infty)$ defined by:

$$h(a,b) = \begin{cases} 2|a| & \text{if } b = 0 \\ |a| + |b| & \text{if } b \neq 0. \end{cases}$$

It can be noted that, $h(a,b) = h(-a,b) = h(a,-b) = h(-a,-b)$. Define g as

$$g(x,y,z) = \max\{h(x-y), h(y-z), h(z-x)\}, \text{ for all } x, y, z \in R_2. \tag{3}$$

Then (R_2, g) is an F_g-metric space. Indeed, (g1) to $(g4)$ are trivial and easy to check. To check $(g5)$, let $P_1 = (a_1, b_1), P_2 = (a_2, b_2), \ldots, P_N = (a_N, b_N) \in R_2$. If $\sum_{i=1}^{N} b_i = 0$, then we have:

$$\begin{aligned} h\left(\sum_{i=1}^{N} P_i\right) = h\left(\sum_{i=1}^{N} a_i, \sum_{i=1}^{N} b_i\right) &= 2\left|\sum_{i=1}^{N} a_i\right| \\ &\leq 2\sum_{i=1}^{N} |a_i| \\ &\leq 2\sum_{i=1}^{N} h(a_i, b_i \\ &= 2\sum_{i=1}^{N} h(pi). \end{aligned}$$

If $\sum_{i=1}^{N} b_i \neq 0$, then we have:

$$
\begin{aligned}
h\left(\sum_{i=1}^{N} P_i\right) = h\left(\sum_{i=1}^{N} a_i, \sum_{i=1}^{N} b_i\right) \quad &= \left|\sum_{i=1}^{N} a_i\right| + \left|\sum_{i=1}^{N} b_i\right| \\
&\leq \sum_{i=1}^{N} |a_i + b_i| \\
&\leq 2\left(\sum_{i=1}^{N} h(a_i, b_i)\right) \\
&= 2\left(\sum_{i=1}^{N} h(pi)\right)
\end{aligned}
$$

From both the cases, we have

$$h(\textstyle\sum_{i=1}^{N} P_i) \leq 2(\sum_{i=1}^{N} h(pi)). \tag{4}$$

Let us fix certain $x, y, z \in R_2$ with $g(x, y, z) > 0$. Let $(u_i)_{i=1}^{m} \subset R_2$, where $m \in \mathbf{N}, m \geq 2$ and $u_1 = x$. We can observe that for any $x, y \in R_2$,

$$
\begin{aligned}
g(x, y, y) &= \max \{h(x-y), h(y-y), h(y-x)\} \qquad (5)\\
&= \max\{h(x-y), h(0), h(y-x)\} \qquad (6)\\
&= h(x-y). \qquad (7)
\end{aligned}
$$

Consider, $g(x, y, z) = \max\{h(x-y), h(y-z), h(z-x)\}$. We have three cases:
Case I: If $g(x, y, z) = h(x-y)$

$$
\begin{aligned}
&= h(x - u_2 + u_2 - u_3 + u_3 - \cdots + u_m - y) \\
&\leq 2\left(\sum_{i=1}^{m-1} h(u_i - u_{i+1})\right) + 2h(u_m - y) \text{ (from "(4)")} \\
&\leq 2\left(\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right) + 2g(u_m, y, z) \text{ (from "(3)" and "(7)")} \\
&= 2\left(\left(\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right) + g(u_m, y, z)\right).
\end{aligned}
$$

Case II: If $g(x, y, z) = h(y-z)$

$$
\begin{aligned}
&= h(z-y) \\
&= h(x - y + z - x) \\
&= h(x - u_2 + u_2 - u_3 + u_3 - \cdots + u_m - y + z - x) \\
&\leq 2\left(\sum_{i=1}^{m-1} h(u_i - u_{i+1})\right) + 2h(u_m - y) + 2h(z - x) \text{ (from "(4)")} \\
&\leq 2\left(\sum_{i=1}^{m-1} h(u_i - u_{i+1})\right) + 2h(u_m - y) + 2h(y - z) \\
&\leq 2\left(\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right) + 4g(u_m, y, z) \text{ (from "(3)" and "(7)")} \\
&= 4\left(\left(\sum_{i=1}^{m-1} g(u_i, u_{i+1}, u_{i+1})\right) + g(u_m, y, z)\right).
\end{aligned}
$$

Case III: If $g(x, y, z) = h(z-x)$

$$
\begin{aligned}
&= h(x-z) \\
&= h(x-y+y-z) \\
&= h(x-u_2+u_2-u_3+u_3-\cdots+u_m-y+y-z) \\
&\leq 2\left(\sum_{i=1}^{m-1} h(u_i-u_{i+1})\right)+2h(u_m-y)+2h(y-z) \ \text{(from "(4)")} \\
&\leq 2\left(\sum_{i=1}^{m-1} g(u_i,u_{i+1},u_{i+1})\right)+4g(u_m,y,z) \ \text{(from "(3)" and "(7)")} \\
&= 4\left(\left(\sum_{i=1}^{m-1} g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right).
\end{aligned}
$$

Therefore, from all three cases, we have

$$
g(x,y,y)=4\left(\left(\sum_{i=1}^{m-1} g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right)
$$

If $g(x,y,z)>0$, then

$$
\log\left(g(x,y,z)\right)\leq \log\left(\left(\sum_{i=1}^{m-1} g(u_i,u_{i+1},u_{i+1})\right)+g(u_m,y,z)\right)+\log\ 4.
$$

So g satisfies (g5) with $f=\log$ and $\alpha=\log\ 4$.
It can be easily verified that $(1,1/n)$ converges to $(1,0)$ in F_g-metric space (R_2,g). But

$$
\begin{aligned}
g((1,1/n),(0,0),(0,0)) &= \max\{h(1,1/n),h(0,0),h(1,1/n)\} \\
&= h(1,1/n) \\
&= 1+1/n \\
&= 1, \text{ as } n\to\infty.
\end{aligned}
$$

And

$$
\begin{aligned}
g((1,0),(0,0),(0,0)) &= \max\{h(1,0),h(0,0),h(1,0)\} \\
&= h(1,0) \\
&= 2(1) \\
&= 2
\end{aligned}
$$

So g is not a jointly continuous function.
Remark 3.10. F_g-metric is not a jointly continuous function.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

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