Compound Weibull Lifetime Distribution-I

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***Abstract* — In this paper a new lifetime distribution named as 'Compound Weibull Lifetime Distribution-I' is derived. The basic assumptions, derivation of the model and some useful characteristics, like mean, variance, distribution function, reliability function, hazard function and cumulative hazard function of the distribution are derived.**

***Keywords* — Hazard function, reliability function, Weibull distribution.**

I. Introduction

Certain trial and error fine-tuning manoeuvres on the part of the producer, leading to certain shifts in process mean (mean of the measurable quality characteristic), assumed to be according to uniform probability law, then the conventional Weibull model will not satisfy for the manufactured product. The present paper emphasizes on cause and effect relationship between quality and reliability of industrial practice. A new lifetime model is derived by compounding the effect of shifts with the lifetime of the product. The properties of this model are also derived.

II. Methodology

Let us assume that:

i. The lifetime (T) of a product follows a Weibull distribution with density f(t), where:

$$f(t)=\begin{cases}\theta\alpha t^{\alpha-1}e^{-\theta t^{\alpha}} & ;\quad 0<t<\infty,\ \theta,\ \alpha>0,\\ 0 & ;\quad elsewhere.\end{cases} \tag{1}$$

ii. Let X be the measurable quality characteristic of the product follows $N(\mu,\sigma^2)$ and let U, L represents the upper and lower specification limits for X, initially defined. Assume that U–L> 6σ, implies that the process is capable of producing better products, meeting the specifications. Suppose the manufacturer decides to relax certain operating characteristics of the process by fine-tuning either one or more among the Material, Men and Machines.

iii. The shift in μ resulting out of fine-tuning follows uniform distribution in $[\mu_L, \mu_U]$, where:

$$\mu U = U-3\sigma and\ \mu L = L+3\sigma \tag{2}$$

It can be observed that the greater the shift in μ which results in a greater deviation from the process mean μ_o will cause in a reduction in the lifetime of the product. This, in turn, results in a reduction in the expected lifetime of the product.

iv. The increase in the absolute deviation of μ from μ_o results in an increase in the reciprocal $v^{\alpha}\Gamma(1+1/\alpha)^{\alpha}$of the expected lifetime and, the same is represented by the relation.

$$\theta^{1/\alpha}\Gamma\left(1+1/\alpha\right)^{-1}=c+mU=V;\ c,\ m>0 \tag{3}$$

where U = | μ – μ_o | which represents absolute deviation of μ from μ_o.

Lemma 2.1: The variation in the random variable U is specified by the probability density function (pdf):

$$g*(u)=\begin{cases}\delta^{-1} & ;\quad 0<u<\delta,\\ 0 & ;\quad \text{elsewhere.}\end{cases} \tag{4}$$

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where $\delta = 2^{-1}(\mu_U - \mu_L)$.

Proof: From the assumption (iii), the pdf of μ is:

$$g(\mu) = \begin{cases} (\mu_U - \mu_L)^{-1} & ; \quad \mu_L < \mu < \mu_U, \\ 0 & ; \quad \text{elsewhere.} \end{cases} \tag{5}$$

Let, $\mu_o = (\mu_U + \mu_L)/2$be the target value. The probability of μ lying between (μ_L, μ_o) is equal to the probability of μ lying between (μ_o, μ_U) and is equal to ½. Then we can obtain:

$$g^*(u)\, du = P(\mu_L < \mu < \mu_o)\frac{du}{\mu_o - \mu_L} + P(\mu_o < \mu < \mu_U)\frac{du}{\mu_U - \mu_o} = \frac{du}{\delta}$$

For $0 < u < \delta$, Hence, the pdf of U is given by:

$$g^*(u) = \begin{cases} \delta^{-1} & ; \quad 0 < u < \delta, \\ 0 & ; \quad elsewhere. \end{cases}$$

III. Findings

Theorem 2.1: The distribution of the lifetime T under the framework as explained above, is given by the compound distribution with density f*(t) as:

$$f^*(t) = \begin{cases} (m\delta)^{-1}\Gamma(1+1/\alpha)^{\alpha}\alpha t^{\alpha-1}\int_c^{c+m\delta} v^{\alpha} e^{-(v\Gamma(1+1/\alpha)t)^{\alpha}} dv & ; \quad 0 < t < \infty, \\ 0 & ; \quad elsewhere. \end{cases} \tag{6}$$

Proof: Using the Lemma 2.1, and using the (3),

$$f_1(v) = \begin{cases} m\delta^{-1} & ; \quad c < v < c + m\delta, \\ 0 & ; \quad elsewhere. \end{cases} \tag{7}$$

then the joint density function of V and T, of a compound distribution denoted by h(v, t) is:

$$h(v, t) = f_1(v) \,.\, f(t \mid v)$$

$$\begin{cases} (m\delta)^{-1} v^{\alpha}\Gamma(1+1/\alpha)^{\alpha}\alpha t^{\alpha-1} e^{-(v\Gamma(1+1/\alpha)t)^{\alpha}} & ; \quad 0<t<\infty \\ 0 & ; \quad elsewhere \end{cases} \tag{8}$$

The marginal density of T can be obtained, by integrating h(v,t) with respect to v, in the appropriate range. Hence the proof.

Note: The pdf f*(t) can be expressed in terms of incomplete gamma distribution function as:

$$f^*(t) = \begin{cases} t^{-2}(m\delta)\Gamma(1+1/\alpha)^{-1}[\Gamma(1/\alpha+1), \omega_1) - \Gamma(1/\alpha+1), \omega_2)] & ; \quad 0 < t < \infty, \\ 0 & ; \quad elsewhere. \end{cases} \tag{9}$$

where

$$\omega_1 = \left[tc\Gamma(1+1/\alpha)\right]^{\alpha} \; and \; \omega_2 = \left[t(c+m\delta)\,\Gamma(1+1/\alpha)\right]^{\alpha} \tag{10}$$

Lemma 2.2: The function f*(t) defined in (6) is a proper probability density function.

Proof: (i) f*(t) should be non-negative, since the integrand in f*(t) is positive as c, m, δ are positive. (ii) Total probability is given by:

$$I \int_0^{\infty} f^*(t)dt = \int_o^{\infty} (m\delta)^{-1}\Gamma(1+1/\alpha)^{\alpha} t^{\alpha-1}\alpha\left[\int_c^{c+m\delta} v^{\alpha} e^{-(v\Gamma(1+1/\alpha)t)^{\alpha}} dv\right]dt = 1$$

By changing the order of integration and by using the transformation:

$$\left(v\,\Gamma(1+1/\alpha)t\right)^{\alpha} = y;\ and\ v^{\alpha}\Gamma(1+1/\alpha)^{\alpha}\ \alpha\ t^{\alpha-1}dt = dy \tag{11}$$

One can establish the above result.

A. Properties Of CWLD – I

Lemma 3.1: The expected value of T for CWLD-l - I is given by:

$$E^*(T) = (m\delta)^{-1}[log\ (1+m\delta c^{-1})] \tag{12}$$

Proof: From the definition of mathematical expectation, one has:

$$E^*(T) = \int_0^{\infty} t\ f^*(t)\ dt = \int_o^{\infty} (m\delta)^{-1}\Gamma(1+1/\alpha)^{\alpha}\ \alpha\ t^{\alpha}\left[\int_c^{c+m\delta} v^{\alpha} e^{-(v\Gamma(1+1/\alpha)t)^{\alpha}} dv\right]dt$$

By changing the order of integration and by using (11):

$$E^*(T) = \int_c^{c+m\delta} (m\delta)^{-1}\left[\int_0^{\infty} e^{-y} y^{1/\alpha} (v\Gamma(1+1/\alpha))^{-1} dy\right]dv = (m\delta)^{-1}[log\ (1+m\delta c^{-1})]$$

Remark: It can be noted that **t**he expected lifetime of the CWLD-I is less than the expected lifetime of Weibull distribution.

Lemma 3.2: The variance of CWLD-I is:

$$V^*(t) = \Gamma(2/\alpha+1)\ \left[c(c+m\delta)\right]^{-1}\left(\Gamma(1+1/\alpha)\right)^{-2} - \left[log\left(1+m\delta c^{-1}\right)\right]^{2}(m\delta)^{-2} \tag{13}$$

Proof: V*(T) = E*(T^2) – (E*(T))^2 and

$$E^*(T^2) = \int_0^{\infty} t^2 f^*(t)dt = \int_0^{\infty} (m\delta)^{-1}\Gamma(1+1/\alpha)^{\alpha}\alpha t^{\alpha+1}\left[\int_c^{c+m\delta} v^{\alpha} e^{-[v\Gamma(1+1/\alpha)t]^{\alpha}} dv\right]dt$$

By changing the order of integration and by using (11):

$$\begin{aligned} E^*(T^2) &= \int_c^{c+m\delta} (m\delta)^{-1}\left[\int_0^{\infty} e^{-y}\ y^{2/\alpha}(v\Gamma(1+1/\alpha))^{-2}dy\right]dv \\ &= \Gamma(2/\alpha+1)\ (m\delta)^{-1}\left(\Gamma(1+1/\alpha)\right)^{-2} m\delta\left[c(c+m\delta)\right]^{-1} \end{aligned} \tag{14}$$

Hence follows.

***Lemma 3.3*:** The distribution function F*(t) of the CWLD-I is given by:

$$h^*(t) = \frac{t^{-2}(m\delta\Gamma(1+1/\alpha)^{-1}[\Gamma(1/\alpha+1,\omega_1)-\Gamma(1/\alpha+1,\omega_2)]}{(m\delta\Gamma(1+1/\alpha)t\alpha)^{-1}[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}$$

Further, using:

$$\Gamma\left(\frac{1}{\alpha}+1,\ \omega_1\right) = 1 - \omega - 1/\alpha + 1/\alpha\Gamma(1/\alpha,\ \omega_1)$$

and

$$\Gamma\left(\frac{1}{\alpha}+1,\ \omega_1\right)=\ 1\ -\omega-\ 1/\alpha\ +\ 1/\alpha\Gamma\left(1/\alpha,\ \omega_2\right)$$

$$h*(t)\ =\frac{\alpha}{t}\frac{\left[\frac{1}{\alpha}\left(\left(\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)\right)+\left(\omega_2^{1/\alpha}e^{-\omega_2}-\omega_1^{1/\alpha}e^{-\omega_1}\right)\right]}{[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}$$

$$=\frac{1}{t}+\frac{\alpha}{t}\frac{\left(\omega_2^{1/\alpha}e^{-\omega_2}-\omega_1^{1/\alpha}e^{-\omega_1}\right)}{[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}$$

Lemma 4.3: The cumulative hazard function H*(t) is given by:

$$H*(t)\ =\int_0^t\left[\frac{1}{x}+\frac{\alpha}{x}\frac{\left(-\omega_1^{1/\alpha}e^{-\omega_1}+\omega_2^{1/\alpha}e^{-\omega_2}\right)}{[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}\right]dx$$

Proof: one has,

$$H*(t)\ =\int_0^t h*(x)\ dx=\int_0^t\left[\frac{1}{x}+\frac{\alpha}{x}\frac{\left(\omega_2^{1/\alpha}e^{-\omega_2}-\omega_1^{1/\alpha}e^{-\omega_1}\right)}{[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}\right]dx$$

IV. CONCLUSION

1. If the manufacturer would like to relax any of the conditions imposed on material, machines and men in a situation, when$U$-$L > 6\sigma$. The Compound Weibull lifetime Distribution can be used rather than the conventional Weibull distribution.
2. The mean and variance of the lifetime of T are: $= (m\delta)^{-1}\left[log\ \left(1+m\delta c^{-1}\right)\right]$ and $V*(t)\ =\ \Gamma\left(2/\alpha+1\right)\ \left[c\left(c+m\delta\right)\right]^{-1}\left(\Gamma\left(1+1/\alpha\right)\right)^{-2}-\ \left[log\left(1+m\delta c^{-1}\right)\right]^{2}\left(m\delta\right)^{-2}$. Its mean lifetime is less than the conventional Weibull distribution.
3. The reliability function is $\left(m\delta\Gamma\left(1+1/\alpha\right)\ t\alpha\right)^{-1}\left[\Gamma\left(1/\alpha,\ \omega_1\right)-\Gamma\left(1/\alpha,\ \omega_2\right)\right]$.
4. The Hazard function of CWLD-I is $t^{-1}+\ \alpha\ t^{-1}\ \frac{[\omega_2^{1/\alpha}e^{-\omega_2}-\omega_1^{1/\alpha}e^{-\omega_1}]}{\left[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)\right]}$ and its cumulative hazard function is $\int_0^t\left[\frac{1}{x}+\frac{\alpha}{x}\frac{\left(-\omega_1^{1/\alpha}e^{-\omega_1}+\omega_2^{1/\alpha}e^{-\omega_2}\right)}{[\Gamma(1/\alpha,\omega_1)-\Gamma(1/\alpha,\omega_2)]}\right]dx$.
5. When $\alpha = 2$ and $\theta = 2^{-1}\ \alpha^{-2}$,the CWLD-I reduces to Compound Rayleigh Lifetime Distribution-I and when $\alpha = 1$, it reduces to Compound Exponential Lifetime Distribution-I.

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CONFLICT OF INTEREST

Authors declare that they do not have any conflict of interest.

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